First excursion probability sensitivity by means of domain decomposition method

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Abstract

This contribution introduces a novel framework for the first excursion probability sensitivity estimation, applicable to linear dynamic systems subject to a Gaussian excitation. The proposed methodology is based on Domain Decomposition method, and the sensitivity estimator is calculated as the partial derivative of the first excursion probability with respect to a design parameter, such as the geometrical dimensions of the system. The linearity of the system plays a key role in building an efficient estimator. Domain Decomposition Method exploits this feature by exploring the failure domain in a very convenient way due to its special structure, characterized by the union of a large number of elementary linear failure domains. This approach allows the sensitivity estimator to be derived as a byproduct of the first excursion probability estimator. The effectiveness of this technique is demonstrated through a numerical example involving a large-scale model.

1 Introduction

The safety level of stochastic structural dynamical systems is of utmost importance for an appropriate design and can be quantified by means of the so-called first excursion probability. This probability indicates if one or more responses of interest exceed a predetermined threshold during a stochastic excitation [1]. The specific case where the system's behavior remains linear (e.g. for serviceability design purposes [2]) and, the stochastic loading is modeled as a Gaussian process, has been addressed through several methods. These methods take advantage of the above-mentioned properties to estimate the first excursion probability in a very efficient way. Some of the well known techniques include a very Efficient Importance Sampling (EIS) [3], the Domain Decomposition Method (DDM) [4], Directional Importance Sampling (DIS) [5] and, multidomain Line Sampling (mLS) [6].

Furthermore, first excursion probabilities can exhibit high sensitivity to perturbations in structural properties, including changes in mass, stiffness, or geometrical dimensions of structural members. Thus, assessing the influence of these parameters on the first excursion probability is crucial for enhancing the reliability analysis [7]. Such information can be very useful in the context of risk evaluation, decision making, as well as reliability-based design optimization problems [8]. Estimating the gradient of failure probability is a complex task that has been addressed in literature (see e.g. [9, 10]), which has been calculated with respect to two different kind of parameters [11]. One group involves the gradient estimation with respect distribution parameters of random variables, which describe uncertain structural properties [12]. The other group, involves the gradient estimation with respect to deterministic parameters related to structural behavior [7, 13], which is the focus of this work.

An approach which is particularly useful to estimate first excursion probabilities is the so-called Domain Decomposition Method [4]. The objective of this contribution is to extend the application of this method towards estimating the sensitivity of the first excursion probability. The focus is on linear structural systems subject to a Gaussian loading, where the sensitivity is calculated with respect to deterministic structural parameters. In this regard, the use of the Domain Decomposition Method plays a key role in the failure domain exploration due to the system's linearity. Furthermore, the sensitivity estimator is developed as a byproduct of the reliability analysis [14], significantly enhancing computational efficiency. Also, a sensitivity analysis of the spectral properties of the system is performed [15].

The remaining sections are organized as follows. Section 2 presents the problem, the first excursion probability, and its gradient definition. Section 3 presents the aforementioned gradient calculation by means of Domain Decomposition Method. Then, one example of the proposed framework is illustrated in Section 4. Finally, Section 5 draws the discussion to a closure and presents thoughts on future developments.

2 Problem statement

2.1 Gaussian loading

The structural system is subjected to a dynamic load p, defined as a discrete Gaussian process over a duration T. This process is discretized into n_T time instants, each of duration Δt, defined as $t_k = (k-1)\Delta t, k = 1, \ldots, n_T$. The dynamic load is represented in terms of the Karhunen-Loève expansion (see e.g. [16, 17]), and is given by:

$$p\left(t_k, \boldsymbol{z}\right) = \mu_k + \boldsymbol{\psi}_k^{\mathrm{T}} \boldsymbol{z}, k = 1, \ldots, n_T, \tag{1}$$

where $p\left(t_k, \boldsymbol{z}\right)$ is the loading at time t_k; $\boldsymbol{z}$ is a realization of a standard Gaussian random variable vector $\boldsymbol{Z}$ of dimensions $n_{KL} \times 1$, with n_{KL} being the order of truncation of the Karhunen-Loève expansion ($n_{KL} \leqslant n_T$); $\boldsymbol{\psi}_k$ is a vector of dimensions $n_{KL} \times 1$ containing information on the covariance of the Gaussian process; and the expected value of the process at time t_k is denoted as μ_k.

2.2 Structural system

The structural system of n_D degrees-of-freedom is assumed as linear elastic with classical damping. It is subjected to the Gaussian loading $p(t, \boldsymbol{z})$, and its equation of motion is given by [18]:

$$\boldsymbol{M}(\boldsymbol{y})\ddot{\boldsymbol{x}}(t, \boldsymbol{y}, \boldsymbol{z}) + \boldsymbol{C}(\boldsymbol{y})\dot{\boldsymbol{x}}(t, \boldsymbol{y}, \boldsymbol{z}) + \boldsymbol{K}(\boldsymbol{y})\boldsymbol{x}(t, \boldsymbol{y}, \boldsymbol{z}) = \boldsymbol{g}(\boldsymbol{y})p(t, \boldsymbol{z}), t \in [0, T], \tag{2}$$

where displacement, velocity, and acceleration are represented by $\boldsymbol{x}$, $\dot{\boldsymbol{x}}$, and $\ddot{\boldsymbol{x}}$, respectively, all vectors of dimension $n_D \times 1$; the mass, damping, and stiffness matrices, $\boldsymbol{M}$, $\boldsymbol{C}$, and $\boldsymbol{K}$, respectively, are of dimensions $n_D \times n_D$; the coupling vector $\boldsymbol{g}$ represents the interaction between the loading and the degrees of freedom of the system, and it has dimensions $n_D \times 1$; and the deterministic vector $\boldsymbol{y}$, containing parameters y_q with $q = 1, \ldots, n_Y$ that represent the structural properties of the system, is of size $n_Y \times 1$. For instance, the vector $\boldsymbol{y}$ can contain parameters related to the geometry of structural members (such as cross-sectional area or length) or material properties (such as Young's Modulus), among others. Importantly, these parameters can be subject to potential changes due to design decisions [19].

The solution of equation (2) allows to control of one or more responses of interest, such as displacements and internal stresses, for example. In this work, the responses of interest correspond to linear combinations of the displacements, denoted as $\eta_i(t, \boldsymbol{y}, \boldsymbol{z}), i = 1, \ldots, n_\eta$, being n_η the number of responses of interest, and are calculated using the convolution integral [18]:

$$\eta_i(t, \boldsymbol{y}, \boldsymbol{z}) = \int_0^t h_i(t-\tau, \boldsymbol{y}) p(\tau, \boldsymbol{z}) d\tau, i = 1, \ldots, n_\eta, \tag{3}$$

where h_i denotes unit impulse response function of the i-th response of interest assuming null initial conditions. This function can be determined through modal analysis [18].

Due to the discretized (in time) definition of the stochastic load in equation (1), it is possible to consider the discretized responses of interest, by means of numerical integration of equation (3), resulting in:

$$\eta_i\left(t_k, \boldsymbol{y}, \boldsymbol{z}\right) = \boldsymbol{a}_{i,k}(\boldsymbol{y})^T \boldsymbol{z}, i = 1, \ldots, n_\eta, k = 1, \ldots, n_T, \tag{4}$$

where $\boldsymbol{a}_{i,k}(\boldsymbol{y})$ is a vector of dimensions $n_{KL} \times 1$, which is defined as:

$$\boldsymbol{a}_{i,k}(\boldsymbol{y}) = \sum_{m=1}^{k} \Delta t \epsilon_m h_i\left(t_k - t_m, \boldsymbol{y}\right) \psi_m, \tag{5}$$

being ϵ_m chosen according to the adopted integration rule [20]. For example, using the trapezoidal rule $\epsilon_m = 1/2$ if $m = 1$ or $m = k$; otherwise $\epsilon_m = 1$.

2.3 First excursion probability

The design requirements are defined in vector $\boldsymbol{b}$ of dimensions $n_\eta \times 1$, where b_i is its i-th element and denotes the prescribed threshold for the response of interest η_i. The performance function $g(\boldsymbol{y}, \boldsymbol{z})$ indicates whether the response of interest η_i exceeds the prescribed threshold b_i during the stochastic excitation, resulting in a negative or positive value, respectively. Thus, the performance function is given by:

$$g(\boldsymbol{y}, \boldsymbol{z}) = 1 - \max_{i=1,\ldots,n_\eta}\left(\max_{k=1,\ldots,n_T}\left(\frac{\left|\eta_i\left(t_k, \boldsymbol{y}, \boldsymbol{z}\right)\right|}{b_i}\right)\right), \tag{6}$$

where $|\cdot|$ is the absolute value and the response of interest η_i is normalized by the threshold b_i. Therefore, the failure domain can be formally defined as $F = \{\boldsymbol{z} \in \mathbb{R}^{n_{KL}} : g(\boldsymbol{y}, \boldsymbol{z}) \leqslant 0\}$.

The probability associated with the failure domain can be quantified by means of the so-called first excursion probability [1]:

$$p_F(\boldsymbol{y}) = \int_{g(\boldsymbol{y},\boldsymbol{z})\leq 0} f_{\boldsymbol{Z}}(\boldsymbol{z}) d\boldsymbol{z}, \tag{7}$$

where $f_{\boldsymbol{Z}(\boldsymbol{z})}$ is the standard Gaussian probability density function in n_{KL} dimensions.

In practical engineering applications, n_{KL} can be in the order of hundreds or thousands. Consequently, the first excursion probability in equation (7) becomes a high-dimensional integral without a closed-form solution, requiring advanced simulation methods [21] for its evaluation. This has led to the development of several methods which leverage the system's linearity to estimate the first excursion probability [3, 4, 5, 6].

2.4 Sensitivity of first excursion probability

The dependence of the failure probability in equation (7) on the vector $\boldsymbol{y}$ suggests that a change in one or more design parameters can impact the value of the failure probability. Specifically, a change in the parameter y_q directly impacts the limit state function (from equation (6), $g(\boldsymbol{y}, \boldsymbol{z}) = 0$). Therefore, studying the sensitivities of the failure probability with respect to different parameters is crucial for design purposes. One potential approach to measure this sensitivity is to calculate the gradient of the first excursion probability [22], as follows:

$$\frac{\partial p_F(\boldsymbol{y})}{\partial y_q} = -\int_{g(\boldsymbol{y},\boldsymbol{z})=0} \frac{\partial g(\boldsymbol{y}, \boldsymbol{z})}{\partial y_q} \frac{1}{\|\nabla_{\boldsymbol{z}} g(\boldsymbol{y}, \boldsymbol{z})\|} f_{\boldsymbol{Z}}(\boldsymbol{z}) dS, q = 1, \ldots, n_Y, \tag{8}$$

where $\|\cdot\|$ denotes Euclidean norm; $\nabla_{\boldsymbol{z}}$ is the nabla operator $\nabla_{\boldsymbol{z}} = [\partial/\partial z_1, \ldots, \partial/\partial z_{n_{KL}}]^T$; and dS denotes a differential element of the limit state hypersurface $S = \{\boldsymbol{z} \in \mathbb{R}^{n_{KL}} : g(\boldsymbol{y}, \boldsymbol{z}) = 0\}$. The equation (8), like equation (7), does not have a closed-form solution, making its estimation a challenging task.

3 Domain Decomposition Method (DDM)

3.1 General remarks

This contribution is based on the Domain Decomposition Method. For this purpose, the failure probability integral is written in terms of the *effective contribution* of each elementary failure domain, obtaining the same first excursion probability estimator that is presented in [4]. Note that the deduction for the Domain Contribution Method presented in here differs from the one originally presented in [4]. Such alternative deduction is chosen on purpose, as it facilitates the calculation of the probability sensitivity.

3.2 Effective contribution of the elementary failure domains

The failure domain defined in Section 2.3 has a unique geometry. It is composed of the union of $n_\eta \times n_T$ elementary failure domains. Each of these elementary failure domains $F_{i,k}$ describes the event where the response η_i exceeds the prescribed threshold b_i at the time instant t_k, and can be decomposed in its positive and negative side, that means $F_{i,k} = F^+_{i,k} \cup F^-_{i,k}$. Then, the elementary failure domain that represents wether the response of interest η_i exceeds its threshold b_i at the time instant t_k is defined as $F^+_{i,k} = \left\{\boldsymbol{z} \in \mathbb{R}^{n_{KL}} : \boldsymbol{a}^T_{i,k}(\boldsymbol{y})\boldsymbol{z} \geqslant b_i\right\}$. Similarly, $F^-_{i,k} = \left\{\boldsymbol{z} \in \mathbb{R}^{n_{KL}} : \boldsymbol{a}^T_{i,k}(\boldsymbol{y})\boldsymbol{z} \leqslant -b_i\right\}$ represents the elementary failure domain that indicates whether the response of interest $-\eta_i$ exceeds its threshold b_i at the time instant t_k. It is straightforward to note that the events $F^+_{i,k}$ and $F^-_{i,k}$ are defined as mutually exclusive events with equal probabilities of occurrence, whenever the mean of the load $\mu_k = 0$, $k = 1, \ldots, n_T$. Now, the failure domain is defined as the union of all the elementary failure domains, that is $F = \cup_{i=1}^{n_\eta} \cup_{k=1}^{n_T} F_{i,k}$. In Figure 1, a schematic representation of the elementary failure domains is shown, for the case where $n_\eta = 1$ and $n_T = n_{KL} = 2$.

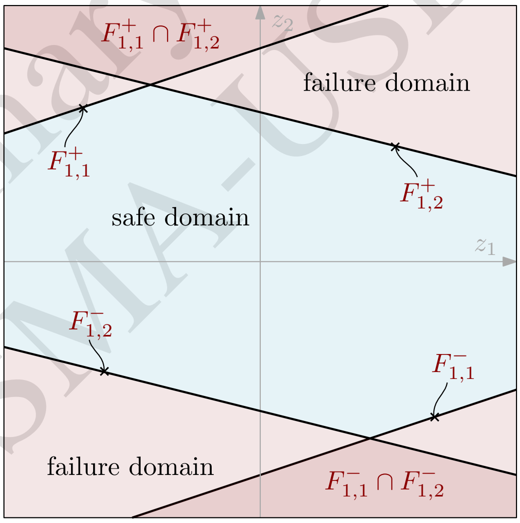

Figure 1: Elementary failure domains representation for the case where $n_\eta = 1$ and $n_T = n_{KL} = 2$.

The existing degree of overlap between elementary failure domains becomes notable when equation (7) involves high dimensions. To address this problem, the analytical definition of the elementary failure domains is crucial, and the first excursion probability of equation (7) can be written in terms of the effective contribution of each of the individual elementary failure domains [3, 4] as:

$$p_F(\boldsymbol{y}) = \sum_{i=1}^{n_\eta} \sum_{k=1}^{n_T} p_{i,k}(\boldsymbol{y}), \tag{9}$$

where $p_{i,k}(\boldsymbol{y})$ is the effective contribution associated with the elementary failure domain $F_{i,k}$, defined as follows:

$$p_{i,k}(\boldsymbol{y}) = \int_{\boldsymbol{z}\in F_{i,k}} \frac{1}{\sum_{h=1}^{n_\eta}\sum_{j=1}^{n_T} I_{F_{h,j}}(\boldsymbol{y},\boldsymbol{z})} f_{\boldsymbol{Z}}(\boldsymbol{z})d\boldsymbol{z}. \tag{10}$$

where $I_{F_{h,j}}(\boldsymbol{y},\boldsymbol{z})$ is an indicator function which is equal to 1 in case that $\boldsymbol{z} \in F_{i,k}$. The *discounting factor* $1/\sum_{h=1}^{n_\eta}\sum_{j=1}^{n_T} I_{F_{i,k}}(\boldsymbol{y},\boldsymbol{z})$ accounts for discounting the effective contribution resulting from the interaction between elementary failure domains. An interpretation of the effective contribution is as follows: if a possible realization $\boldsymbol{z}$ of $\boldsymbol{Z}$ belongs to two or more elementary failure domains, the discounting factor is less than 1. This implies that the effective contribution $p_{i,k}$ corresponds to the probability of occurrence of the event $F_{i,k}$, reduced by the discounting factor due to the overlap between elementary failure domains.

3.3 First excursion probability by means of Domain Decomposition Method

The estimation of the failure probability shown in equation (9) is done by estimating the effective contribution of the elementary failures domains. In order to achieve this, the equation (10) is written using the Directional Sampling scheme [23]. This technique allows writing the realization vector $\boldsymbol{z}$ in terms of its Euclidean norm r and its unit direction $\boldsymbol{u}$, that means $\boldsymbol{z} = r\boldsymbol{u}$. The unit vector is defined in the standard Gaussian space and is calculated as $\boldsymbol{u} = \boldsymbol{z}/\|\boldsymbol{z}\|$, and the Euclidean norm is defined as $r = \|\boldsymbol{z}\|$, where r^2 follows a Chi-squared distribution of n_{KL} degrees-of-freedom [24]. Therefore, the resulting effective contribution integral is reformulated as:

$$p_{i,k}(\boldsymbol{y}) = \int_{\boldsymbol{u}\in\Omega_{\boldsymbol{U}}} \int_{r\boldsymbol{u}\in F_{i,k}} \frac{2r f_{R^2}\left(r^2\right) f_{\boldsymbol{U}}(\boldsymbol{u})}{\sum_{h=1}^{n_\eta}\sum_{j=1}^{n_T} I_{F_{h,j}}(\boldsymbol{y},r\boldsymbol{u})} dr d\boldsymbol{u}, \tag{11}$$

where $\Omega_{\boldsymbol{U}} = \left\{\boldsymbol{u} \in \mathbb{R}^{n_{KL}} : \boldsymbol{u}^T\boldsymbol{u} = 1\right\}$ denotes the sample space for $\boldsymbol{u}$; $f_{\boldsymbol{U}}(\boldsymbol{u})$ corresponds to the uniform probability density function over the $(n_{KL}-1)$-dimensional hypersphere; and $f_{R^2}(\cdot)$ is the Chi-squared probability density function with n_{KL} degrees of freedom. It is possible to demonstrate [24] that the term $2rf_{R^2}(r^2)$ arises from transforming the probability distribution associated with r to the Chi-squared probability distribution, which depends on r^2.

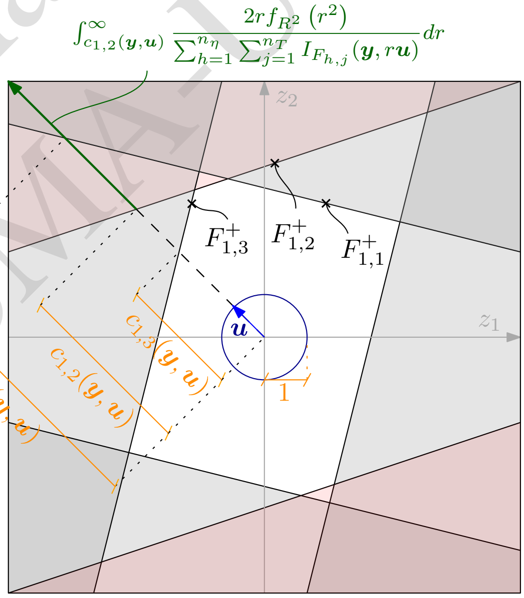

Figure 2: Inner integral of equation (11) in the context of $p_{1,2}$ estimation for the case where $n_\eta = 1$, $n_T = 3$ and $n_{KL} = 2$.

For a better understanding of the effective contribution in the context of Directional Sampling, Figure 2 illustrates the case with $n_\eta = 1$, $n_T = 3$, and $n_{KL} = 2$ when estimating $p_{1,2}$. For simplicity, only the positive side of the elementary failure domains are labeled. It is worth noting that, from equation (11), the inner integral (highlighted with the green arrow in Figure 2) given a realization of the unit direction vector $\boldsymbol{u}$, has an analytical solution due to the system's linearity. Indeed, it can be solved by decomposing the integration interval $[c_{1,2}(\boldsymbol{y},\boldsymbol{u}),\infty[$ into segments, where in each of these segments, exhibits a different degree of overlap between elementary failure domains. In other words, the integration interval is subdivided into parts where the discounting factor $1/\sum_{h=1}^{n_\eta}\sum_{j=1}^{n_T} I_{F_{h,j}}(\boldsymbol{y},r\boldsymbol{u})$ from equation (11) remains constant. For instance, in Figure 2, the values of the discounting factor for each segment are presented in Table 1:

Table 1: Effective contribution discounting factor in example shown in Figure 2.

segment	$1/\sum_{h=1}^{n_\eta}\sum_{j=1}^{n_T} I_{F_{h,j}}(\boldsymbol{y},r\boldsymbol{u})$
$[c_{1,3}(\boldsymbol{y},\boldsymbol{u}),c_{1,2}(\boldsymbol{y},\boldsymbol{u})[$	-
$[c_{1,2}(\boldsymbol{y},\boldsymbol{u}),c_{1,1}(\boldsymbol{y},\boldsymbol{u})[$	2
$[c_{1,1}(\boldsymbol{y},\boldsymbol{u}),\infty[$	3

Note that even though the failure domain includes the event $F_{1,3}$, the integration is performed within the domain of $F_{1,2}$ when estimating the effective contribution $p_{1,2}$.

The calculation of a single effective contribution also involves solving the outer integral of equation (11). This integration requires significant computational effort due to the evaluation of all possible directions $\boldsymbol{u}$. However, the effective contribution can be estimated efficiently by introducing an importance sampling density function $f_{\boldsymbol{U}}^{\mathrm{IS},(i,k)}(\boldsymbol{u})$ into equation (11). The importance sampling density function is based on the ideas proposed in [3, 5, 25], with the difference that each effective contribution $p_{i,k}$ has its own importance sampling density function, given by:

$$f_{\boldsymbol{U}}^{\mathrm{IS},(i,k)}(\boldsymbol{u}) = f_{\boldsymbol{U}}\left(\boldsymbol{u}|F_{i,k}\right), \tag{12}$$

where $f_{\boldsymbol{U}}\left(\boldsymbol{u}|F_{i,k}\right)$ is the probability density associated with the direction $\boldsymbol{u}$ conditioned on the occurrence of an elementary failure event $F_{i,k}$.

In practical implementation, explicitly calculating all efective contributions $p_{i,k}$ shown in equation (9), involves a considerable computational effort. Nevertheless, the summation of the effective contribution terms can be estimated through simulation [4] by incorporating weights that can be interpreted as a probability mass function. A very convenient way to define them is to be proportional to the probability of occurrence of the event $F_{i.k}$ (as in [3]). Therefore, equation (9) can be written as:

$$p_F(\boldsymbol{y}) = \sum_{i=1}^{n_\eta}\sum_{k=1}^{n_T}\left(\frac{1}{w_{i,k}}p_{i,k}(\boldsymbol{y})\right)w_{i,k}, \tag{13}$$

where the weight $w_{i,k}$ is defined as follows:

$$w_{i,k} = \frac{P\left[F_{i,k}\right]}{\sum_{l=1}^{n_\eta}\sum_{m=1}^{n_T} P\left[F_{l,m}\right]}. \tag{14}$$

The expression in equation (13) involves a summation over a discrete random variable $w_{i,k}$ and an integration over a continuous random variable $\boldsymbol{u}$. This can be solved through simulation by generating samples of both random variables, as follows:

$$p_F \approx \tilde{p}_F(\boldsymbol{y}) = \frac{1}{N}\sum_{j=1}^{N}\left(\frac{1}{w_{(i,k)^{(j)}}}\tilde{p}_{(i,k)^{(j)}}(\boldsymbol{y},\boldsymbol{u}_{(i,k)^{(j)}})\right), \tag{15}$$

where the pair (i,k) associated with the sample (j) is randomly selected with probability proportional to the weights $w_{i,k}$; N is the total number of samples; the vector $\boldsymbol{u}_{(i,k)^{(j)}}$ is distributed according to $f_{\boldsymbol{U}}^{\mathrm{IS},(i,k)}(\boldsymbol{u})$; and $\tilde{p}_{(i,k)^{(j)}}(\boldsymbol{y},\boldsymbol{u}_{(i,k)^{(j)}})$ is a single estimation of the effective contribution $p_{i,k}(\boldsymbol{y})$, where $\boldsymbol{u}_{(i,k)^{(j)}}$ denotes a sample drawn from $f_{\boldsymbol{U}}^{\mathrm{IS},(i,k)}(\boldsymbol{u})$. Thus, equation (15) is the first excursion probability estimator by means of

Domain Decomposition Method.

3.4 Sensitivity of first excursion probability by means of Domain Decomposition Method

The sensitivity of the first excursion probability quantifies changes in the failure probability resulting from potential modifications in design parameters. These changes directly impact the limit state hypersurface associated with the failure domain. A geometrical representation of this situation is shown in Figure 3, for the case where $n_\eta = n_Y = 1$ and $n_T = n_{KL} = 2$. The failure domain is represented by the region $g(y_q, \boldsymbol{z}) \leqslant 0$, and the safe domain is represented by the region $g(y_q, \boldsymbol{z}) > 0$. The limit state function $g(y_q, \boldsymbol{z}) = 0$ is perturbed due to a change Δ in the design parameter y_q, resulting in $g(y_q + \Delta, \boldsymbol{z}) = 0$.

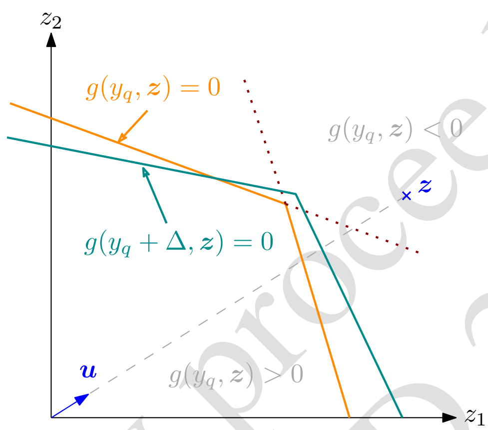

Figure 3: Sensitivity representation of the limit state function for the case where $n_\eta = n_Y = 1$ and $n_T = n_{KL} = 2$ due to a perturbation in the design parameter y_q.

Clearly, while exploring a possible realization $\boldsymbol{u}$ drawn from $f_{\boldsymbol{U}}^{\text{IS},(i,k)}(\boldsymbol{u})$ (as defined in equation (12)) within the framework of the Domain Decomposition Method, the effective contribution $p_{i,k}$ is affected by the design parameter perturbation. Consequently, the failure probability estimator changes.

The desired sensitivities can be obtained by calculating the partial derivative of equation (13) with respect to a design parameter $y_q = 1, \ldots, n_Y$, leading to:

$$\frac{\partial p_F(\boldsymbol{y})}{\partial y_q} = \sum_{i=1}^{n_\eta} \sum_{k=1}^{n_T} \left(\frac{1}{w_{i,k}} \frac{\partial p_{i,k}(\boldsymbol{y})}{\partial y_q} \right) w_{i,k}, \tag{16}$$

where the term $\partial p_{i,k}(\boldsymbol{y})/\partial y_q$ can be calculated using Leibniz' rule [22]. The chain rule for differentiation involves calculating the derivatives of the discretized responses of interest from equation (3). This is followed by calculating the derivatives of the impulse response functions and, consequently, the derivatives of the system's spectral properties (natural frequencies and mode shapes). This task can be efficiently performed using the method proposed in [15], which involves solving a system of linear equations associated with the system's stiffness and mass matrices, as well as their derivatives.

The sensitivity of the first excursion probability in equation (16) can also be estimated through simulation by sampling the discrete random variable $w_{i,k}$ and the continuous random variable $\boldsymbol{u}$, resulting in:

$$\frac{\partial p_F(\boldsymbol{y})}{\partial y_q} \approx \frac{\partial \tilde{p}_F(\boldsymbol{y})}{\partial y_q} = \frac{1}{N} \sum_{j=1}^{N} \left(\frac{1}{w_{(i,k)^{(j)}}} \frac{\partial \tilde{p}_{(i,k)^{(j)}} \left(\boldsymbol{u}_{(i,k)^{(j)}} \right)}{\partial y_q} \right), \tag{17}$$

where the pair (i, k) associated with the sample (j) is randomly selected with probability proportional to the weights $w_{i,k}$; N is the total number of samples; the vector $\boldsymbol{u}_{(i,k)^{(j)}}$ is distributed according to $f_{\boldsymbol{U}}^{\text{IS},(i,k)}(\boldsymbol{u})$; and

$\partial\tilde{p}_{(i,k)^{(j)}}(\boldsymbol{y}, \boldsymbol{u}_{(i,k)^{(j)}})/\partial y_q$ is a single estimation of the derivative of the effective contribution $\partial p_{i,k}(\boldsymbol{y})/\partial y_q$, where $\boldsymbol{u}_{(i,k)^{(j)}}$ denotes a sample drawn from $f_{\boldsymbol{U}}^{\mathrm{IS},(i,k)}(\boldsymbol{u})$. Therefore, equation (17) is the gradient of the first excursion probability estimator, with respect to the design parameter y_q, by means of Domain Decomposition Method.

4 Numerical example

The gradient estimates of the first excursion probability, by means of Domain Decomposition Method, are presented in a numerical example involving a three-dimensional finite element model which comprises 29466 degrees-of-freedom, where linear elastic behavior is assumed. The model, illustrated in Figure 4, is a 16-story reinforced concrete building where each typical floor involves columns, beams, slab and a shear wall core (highlighted as blue in Figure 4). The slabs of each floor have a thickness of 18 [cm], while the shear walls possess a nominal thickness of 40 [cm], both modeled using shell elements. The interstory height is equal to 3.24 [m], which gives a total building height of 52.0 [m]. The material properties for the reinforced concrete are giving by the Young's modulus $E = 2.5 \times 10^{10}$ [N/m^2], the Poisson ratio $\nu = 0.3$ and the mass density equal to 2500 [kgf/m^3]. To ensure a correct characterization of the response, the first 20 mode shapes are retained. In addition, a classical damping of 5% is considered for all modes.

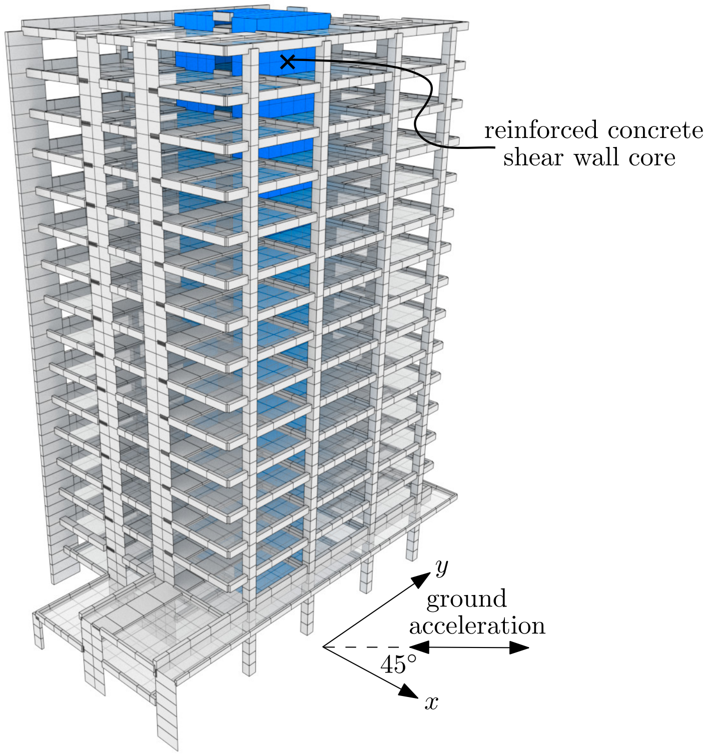

Figure 4: Perspective view of schematic representation of 16-story reinforced concrete building subject to stochastic ground acceleration.

The stochastic ground acceleration is modeled as a modulated discrete white noise process applied at 45° with respect to the x axis. This white noise passes through a Clough-Penzien filter (see e.g. [26]) and has a spectral density of $S = 3 \times 10^{-3}$ [m^2/s^2]. The duration of the ground acceleration is $T = 10$ [s], discretized into 1001 time instants ($n_{KL} = n_T = 1001$), each of duration $\Delta = 0.01$ [s]. The discrete white noise process is modulated by the following function $m(t)$:

$$m(t) = \begin{cases} (t/5)^2 & 0 \leq t \leq 5[\,\mathrm{s}] \\ 1 & 5 \leq t \leq 6[\,\mathrm{s}] \\ e^{-(t-6)^2} & t > 6[\,\mathrm{s}] \end{cases} . \tag{18}$$

In addition, the Clough-Penzien filter is characterized by the natural circular frequencies $\omega_{g,1} = 15.6$ [rad/s] and $\omega_{g,2} = 10$ [rad/s], and the damping ratios $\zeta_{g,1} = 0.6$ and $\zeta_{g,2} = 0.9$.

The sixteen interstory drifts corresponds to the responses of interest in both x and y directions. They are measured with respect to the center of each of the floors during the stochastic excitation. That means a total of $n_\eta = 32$ responses of interest in 1001 time instants, being involved a total of 32000 elementary failure domains. The performance criterion is based on serviceability requirements and specifies that none of the responses should exceed a prescribed threshold of $b_i = 6.5$ mm, which is 0.2% of the floor height. The first excursion probability associated with this problem is calculated by means of Domain Decomposition Method, resulting in $\tilde{p}_F \approx 2.0 \times 10^{-3}$. The objective is to estimate the sensitivity of the first excursion probability by means of Domain Decomposition Method with respect to the vector $\boldsymbol{y} = [y_1, \ldots, y_8]^T$, where y_q denotes the thickness of the reinforced concrete shear wall core of the $(2q-1)$-th and (2q)-th floors. For instance, y_2 represents the thickness of the shear walls from the reinforced concrete core on the 3rd and 4th floors.

The sensitivity estimates were calculated as a byproduct of the reliability analysis, offering a significant advantage in terms of computational effort. Furthermore, it is important to note that the results are presented based on the number of samples, with each sample involving one dynamic analysis and one sensitivity analysis.

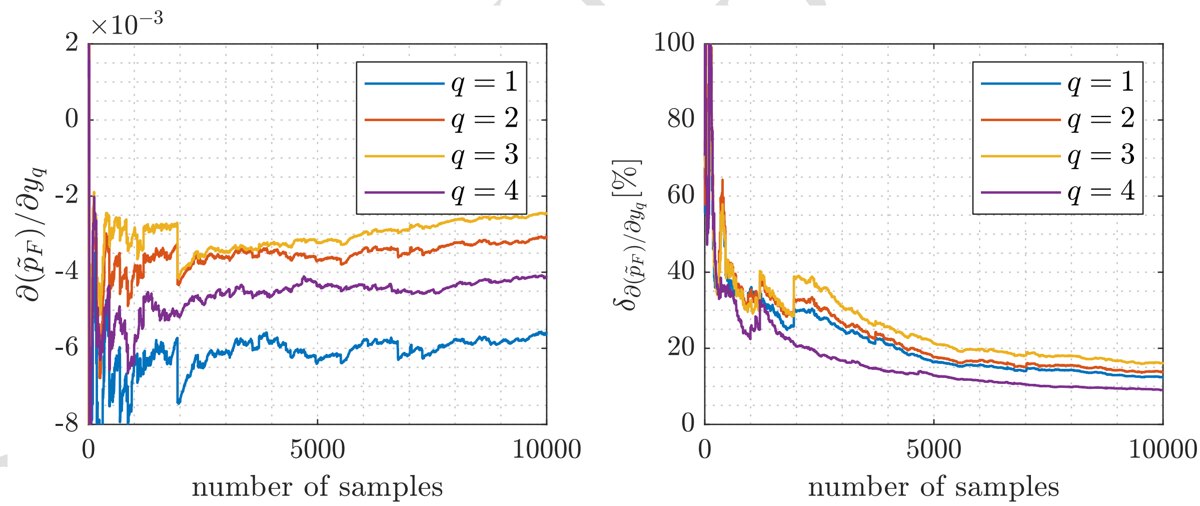

Figure 5: Evolution of the estimator of the partial derivative of the probability with respect to $y_q, q = 1, 2, 3, 4$ and its coefficient of variation with respect to the number of samples, using DDM.

The results of the sensitivity estimates are presented in Figures 5 and 6. The left side of the first figure shows the evolution of the sensitivity estimates associated with the design parameters y_q, for $q = 1, 2, 3, 4$ with respect to the number of samples, while the right side shows the evolution of the coefficient of variation associated with the same estimates with respect to the number of samples. The second figure shows the same situation as before, but the curves are associated with the remaining design parameters y_q, for $q = 5, 6, 7, 8$. It is possible to observe that stable estimates can be obtained with a reduced number of samples. The authors have verified the performance compared with Monte Carlo simulation (using finite differences), which requires generating a large number of samples of the order of millions, to achieve stable estimates, while DDM requires approximate 5000 samples for most estimators. Monte Carlo Simulation results are not shown for the sake of brevity. Furthermore, the results presented in Figures 5 and 6 indicates that for

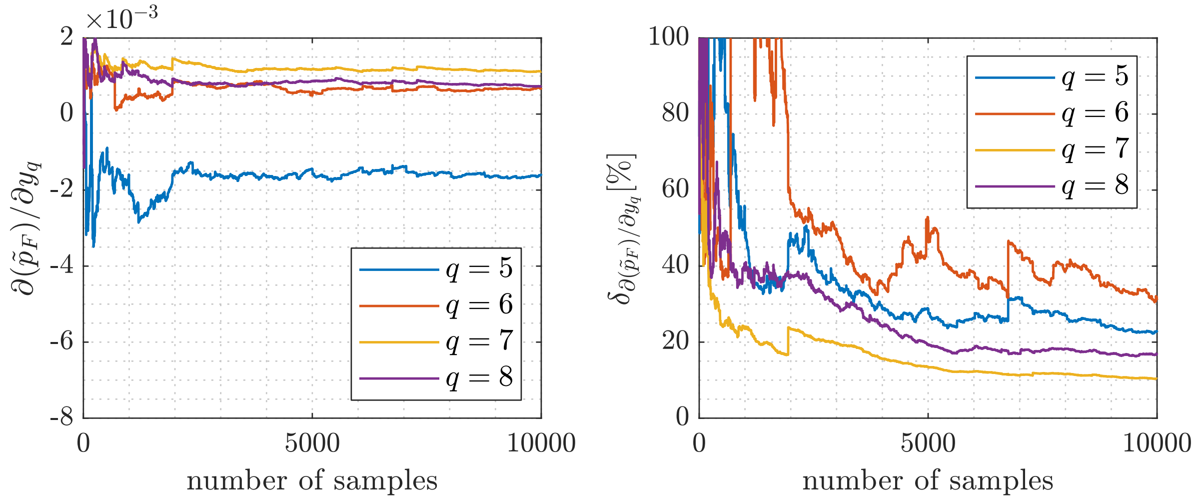

Figure 6: Evolution of the estimator of the partial derivative of the probability with respect to $y_q, q = 5, 6, 7, 8$ and its coefficient of variation with respect to the number of samples, using DDM.

$q = 1, 2, 3, 4, 5$, the sensitivity estimates are negative. The interpretation is that increasing the shear wall's thickness from the first floor to the tenth floor results in a decrease in the failure probability. Hence, it reduces the maximum displacements of the building. Then, for $q = 6, 7, 8$, the sensitivity estimates are positive. The latter means that increasing the shear wall's thickness from the eleventh to the sixteenth floor, increases the failure probability. This occurs due to a stiffening in the upper floors, which tends to behaves as a rigid body, and as a consequence, produces an increasing in the maximum displacements of the building. While some results may seem intuitive, it is important to note that the magnitude associated with y_4 is larger than y_1, y_2, and y_3. These conclusions highlight the complexity of the influence of design parameters on the failure domain and emphasize the importance of studying sensitivities.

5 Conclusions

This contribution has explored the application of Domain Decomposition Method for estimating the sensitivity of the first excursion probability of a linear system subject to a Gaussian loading. The sensitivity corresponds to the partial derivative with respect to design parameters that affect the structural response. The calculation of the sought sensitivities is achieved with a reduced number of samples, demonstrating high efficiency and stability. The proposed framework gathers vauable information of the failure domain, by exploring the failure domain in a directional way. For each line explored, the information of the effective contribution of the failure probability and its gradient for each elementary failure domain is incorporated into both estimators. Furthermore, the sensitivities are estimated as a byproduct of the reliability analysis.

Future extensions of the presented research could explore:

- An extension to more general types of Gaussian excitation.
- Designing a modified Importance Sampling Density function, for efficiency purposes.
- The sensitivity calculation with respect to excitation parameters, i.e., frequencies of the Clough-Penzien model filters.
- Application of the framework in the context of reliability based design optimization (RBO) problems.

The above-mentioned issues are currently being investigated by the authors.

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