A reduced-order model for interval analysis in linear dynamical systems

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Abstract
Information about properties of linear dynamical systems is often unavailable as precise values. Instead, their characterization may be affected by imprecision, vagueness, ambiguity, or even incompleteness. In such a case, the system and its response are affected by epistemic uncertainty. To address this uncertainty, set-based methods such as interval analysis have been considered. Nevertheless, the associated numerical costs may increase considerably when interval methods are used. For dynamical systems, this numerical cost arises from the need for repeated eigenvalue/eigenvector analyses. Given this challenge, this paper proposes a framework for performing interval analysis for problems of linear elastic dynamics, considering a Reduced-Order Model (ROM). This ROM projects the dynamic equilibrium equations into a small-dimensional basis. This basis consists of a subset of the normalized modal shapes of the system obtained for crisp reference values associated with the interval quantities. The proposed strategy is tested in a truss structure.

1 Introduction

The study of dynamical systems can be significantly affected by a large number of uncertainties, making the accurate prediction of system response a challenging task. When these uncertainties arise due to data scarcity, lack of information, or vagueness, the system properties are considered to be affected by epistemic uncertainty. Interval analysis techniques [1] have become a useful tool for quantifying epistemic uncertainty in engineering applications. These methods allow users to assess the potential deviations in system parameters and translate this information into variations in the response of interest. However, when traditional interval analysis approaches are applied to finite elements (FE), computational demands increase significantly, particularly for complex systems involving dynamic analysis. Therefore, this work addresses the challenge of efficiently propagating uncertainty related to interval analysis. Specifically, it focuses on linear dynamical systems.

Propagating uncertainty from input parameters to the response is nontrivial when using interval input parameters. Repeated deterministic analyses are necessary for different input parameter realizations to obtain the interval response [2]. Given this challenge, the goal of this paper is to reduce the computational cost of interval response estimation. To effectively deal with this difficulty, a surrogate model corresponding to a Reduced-Order Model (ROM) is considered to replace the original FE model [3]. The advantage of the surrogate model lies in its ability to map the system equations onto a lower dimensional basis, which significantly reduces the time required to compute the response.

The use of such models has been widely extended to the study of dynamical systems under various types of uncertainty. Researchers have focused on approximating eigenfrequencies and mode shape vectors to evaluate frequency response functions (FRF) [4]. For example, Pradlwarter et al. (2002) [5] addressed the computational challenges of solving eigenvalue problems in large systems with random properties and presented methods to reduce the complexity of Monte Carlo simulations. Similarly, Goller et al. (2011) [6] proposed a meta-model using interpolation between matrices at support points that effectively handle closely spaced eigenfrequencies and mode switching. Another type of approximate model was introduced by Sim et al. 2007 [7], who proposed an approach to computing the FRF by considering exact bounds on the natural frequencies and a first-order Taylor expansion of the mode shapes. De Gersem et al. (2005) [8] extended the Modal Rectangle (MR) method, which translates deterministic modal superposition into interval analysis, with the Modal Rectangle with Eigenvalue Interval Correction (MRE) method to account for eigenvalue variations. The study of Giannini and Hanss (2008) [9] employs a model dimension reduction by component mode synthesis. The work of Beer and Liebscher (2008) [10] presented a general framework for design under fuzziness that combines cluster analysis, an appropriate optimization strategy, and robustness evaluation. On the other hand, Valdebenito et al. 2016 [11] proposed a meta-model to approximate the frequency response function (FRF) through two levels of approximation: spectral properties and linear expansions of mode shapes and natural frequencies. Other notable contributions include van Mierlo et al. (2022) [12], who used adaptive kriging for interval field techniques to model uncertain boundary conditions in crash simulations. More recently, Callens et al. (2022) [13] evaluated the performance of the multilevel quasi-Monte Carlo (MLQMC) technique for interval analysis in terms of accuracy and computational cost for a nonlinear input-output relationship.

The strategy proposed in this work also benefits from the use of the FRF to obtain the uncertain response, in particular by seeking an approximate representation of it. The reduced basis is especially suitable because its construction requires only three exact analyses. In this work, this basis comprises a subset of the normalized modal shapes of the system obtained for crisp reference values associated with the interval variables. The proposed strategy is tested for estimating the uncertain maximum frequency response function (FRF) in a 210-degree-of-freedom truss system due to the addition of two uncertain masses, where accurate estimation of the response extremes is possible with reduced numerical effort.

2 Formulation of the problem

2.1 Governing equations

The focus of this work is the study of linear dynamical systems which are analyzed using the frequency response function (FRF). Consider that the structural properties of the system are difficult to determine precisely due to problems such as lack of knowledge, imprecision, vagueness, and scarcity of data. As a result, the input parameters are subject to epistemic uncertainty and are represented as a vector $\boldsymbol{\xi}$ of size n_ξ. The structural system is modeled using the finite element (FE) method and is characterized considering a total of n_d degrees-of-freedom (DOF).

The equation of motion that relates the structural properties, load, displacement, velocity, and acceleration of the FE model of the structure is as follows

$$\boldsymbol{M}(\boldsymbol{\xi})\ddot{\boldsymbol{u}}(t,\boldsymbol{\xi}) + \boldsymbol{C}(\boldsymbol{\xi})\dot{\boldsymbol{u}}(t,\boldsymbol{\xi}) + \boldsymbol{K}(\boldsymbol{\xi})\boldsymbol{u}(t,\boldsymbol{\xi}) = \boldsymbol{f}(t), \tag{1}$$

where t represents the time; $\boldsymbol{M}(\boldsymbol{\xi}) \in \mathbb{R}^{n_d \times n_d}, \boldsymbol{C}(\boldsymbol{\xi}) \in \mathbb{R}^{n_d \times n_d}$ and $\boldsymbol{K}(\boldsymbol{\xi}) \in \mathbb{R}^{n_d \times n_d}$ are the mass, damping and stiffness matrices, respectively; $\ddot{\boldsymbol{u}}(t,\boldsymbol{\xi}) \in \mathbb{R}^{n_d \times 1}$, $\dot{\boldsymbol{u}}(t,\boldsymbol{\xi}) \in \mathbb{R}^{n_d \times 1}$ and $\boldsymbol{u}(t,\boldsymbol{\xi}) \in \mathbb{R}^{n_d \times 1}$ are the acceleration, velocity and displacement of the structural system, respectively; and $\boldsymbol{f}(t) \in \mathbb{R}^{n_d \times 1}$ is the load vector. Note that it is assumed that $\boldsymbol{C}(\boldsymbol{\xi})$ corresponds to classical damping. It is observed that the system's matrices $\boldsymbol{M}(\boldsymbol{\xi}), \boldsymbol{C}(\boldsymbol{\xi})$ and $\boldsymbol{K}(\boldsymbol{\xi})$, depend on the uncertain input variables $\boldsymbol{\xi}$, while the external loads $\boldsymbol{f}(t)$ are assumed to be unaffected by the uncertainty. Therefore, the system's response also depends on these uncertain input parameters. It should be noted that the number of degrees-of-freedom n_d needed to represent the system using the FE method is usually high. Thus, the solution of eq. 1 involves a significant computational effort [11]. The next subsection discusses how to compute this response.

2.2 Structural response

Typically, designers are concerned with finding a particular response of the structural system $r(\boldsymbol{\xi})$ (e.g., maximum displacement). One way to obtain this response is to use the so-called frequency response function (FRF) [14]:

$$\boldsymbol{H}(\omega, \boldsymbol{\xi}) = \sum_{m=1}^{n_m} \frac{\boldsymbol{\phi}_m(\boldsymbol{\xi})\boldsymbol{\phi}_m^T(\boldsymbol{\xi})}{\omega_m(\boldsymbol{\xi})^2 - \omega^2 + i2d_m\omega\omega_m(\boldsymbol{\xi})}, \tag{2}$$

where $\boldsymbol{H}(\omega, \boldsymbol{\xi})$ has dimension $n_d \times n_d$, n_m is the number of modes retained for dynamic analysis ($n_m \leq n_d$), $\omega_m(\boldsymbol{\xi})$ and $\boldsymbol{\phi}_m(\boldsymbol{\xi})$ denote the $m-$th natural frequency and the $m-$th mode shape, respectively; while d_m denotes the damping ratio associated with the $m-$th mode. It is assumed that d_m does not depend on $\boldsymbol{\xi}$.

From eq. 2 it is clear that the major task is calculating the spectral properties (i.e., to find the natural frequencies and mode shapes). In this context, the natural frequencies and mode shapes are calculated through a dynamic structural analysis, which involves solving the following eigenvalue/eigenvector problem

$$\left(\boldsymbol{K}(\boldsymbol{\xi}) - \omega_m^2(\boldsymbol{\xi})\boldsymbol{M}(\boldsymbol{\xi})\right)\boldsymbol{\phi}_m(\boldsymbol{\xi}) = 0, \quad m = 1, \ldots, n_m. \tag{3}$$

It is noteworthy that these mode shapes constitute a basis that contains information on the system behavior. Solving eq. 3 can be numerically demanding, especially for structures with a large number of degree-of-freedom n_d and subject to uncertainty. Therefore, two tasks remain: how to incorporate the epistemic uncertainty and how to avoid the numerical cost associated with solving the eigenvalue and eigenvector problem. For the first task, and following a set-based approach, the definition of intervals is a straightforward way to describe the epistemic uncertainty of the input parameters collected in the vector $\boldsymbol{\xi}$. The essential definitions for incorporating this uncertainty using interval analysis are discussed in Section 3. For the second task, Section 4 discusses the application of a surrogate model.

3 Interval analysis

3.1 Interval variables

In the initial design phase, data relating to mass, damping, and stiffness are often subject to epistemic uncertainty. Unfortunately, these data alone are not sufficient to construct a robust numerical model for predicting the behavior of a structure. To address this challenge, one approach is to use interval analysis [1]. An interval or interval scalar is a convex subset of the domain of real numbers $\mathbb{R}$. An interval parameter ξ^I is defined by

$$\xi^I = [\underline{\xi}, \bar{\xi}] = \{\xi \in \mathbb{R} \mid \underline{\xi} \leq \xi \leq \bar{\xi}\}, \tag{4}$$

where $\underline{\xi}$ is the lower bound and $\bar{\xi}$ is the upper bound of ξ^I. Note that by definition ξ^I contains all possible values that an uncertain input parameter can assume, without making any assumptions about the likelihood of those values [15].

Most dynamic analyses involve multiple uncertainties. In this context, the definition of an interval vector is convenient. An interval vector $\boldsymbol{\xi}^I$ is a vector where each element is an interval

$$\boldsymbol{\xi}^I = \left\{ \begin{array}{c} \xi_1^I \\ \xi_2^I \\ \vdots \\ \xi_n^I \end{array} \right\} = \left\{ \boldsymbol{\xi} \in \mathbb{R}^n \mid \xi_i \in \xi_i^I \right\}, \tag{5}$$

with $\boldsymbol{\xi}^I \in \mathbb{IR}^n$, the domain of closed real-valued interval vectors of size n. Consequently, an n-dimensional interval vector describes a hypercube in n-dimensional space. The lower and upper bounds of the interval scalar entries in the interval vector $\boldsymbol{\xi}^I$ of eq. 5 determine the vertices of this hypercube [2]. The following section discusses how to find these limits.

3.2 Interval response

The main principle of interval analysis is to search, from a hypercube $\boldsymbol{\xi}^I$, for those parameter realizations that lead to the extreme response of the system. In the case that the response of interest is scalar r^I, e.g. maximum FRF at a DOF i due to a force applied in the DOF j, the optimization problem corresponds to

$$\underline{r} = \min_{\boldsymbol{\xi}\in\boldsymbol{\xi}^I} r(\omega,\boldsymbol{\xi}) = \min_{\boldsymbol{\xi}\in\boldsymbol{\xi}^I}\left(\max_{\omega\in\Omega}(|H_{i,j}(\omega,\boldsymbol{\xi})|)\right) \tag{6}$$

$$\bar{r} = \max_{\boldsymbol{\xi}\in\boldsymbol{\xi}^I} r(\omega,\boldsymbol{\xi}) = \max_{\boldsymbol{\xi}\in\boldsymbol{\xi}^I}\left(\max_{\omega\in\Omega}(|H_{i,j}(\omega,\boldsymbol{\xi})|)\right), \tag{7}$$

where $r^I = [\underline{r},\bar{r}]$ is the interval response of the system which is defined by its lower $\underline{r}$ and upper $\bar{r}$ bounds. Note that since it is specified that the FRF approach is used to determine the response of interest, it also depends on ω. A global optimization approach is commonly used to compute the bounds associated with r^I. Nevertheless, using a global optimization approach to compute response bounds requires repeated deterministic analysis by examining different realizations of the uncertain input parameters. Thus, the objective of this work is to reduce the numerical cost of estimating the interval response. To effectively deal with this challenge, a surrogate model corresponding to a Reduced-Order Model (ROM) is considered to replace the original finite element model. The details of the construction of this approximate model are discussed in the next section.

4 Reduced-order model

In the previous section, the interval technique was reviewed for incorporating uncertainty in input parameters into a finite element (FE) model when modeling a dynamical system. Once the uncertainty is described using intervals, it becomes necessary to propagate it through the response. However, this propagation can be computationally expensive. To overcome this obstacle, one can employ a surrogate model, such as a Reduced-Order Model (ROM). The ROM allows the equilibrium equations of the entire system to be projected onto a lower-dimensional basis.

The reduced-order model considered in this contribution takes advantage of modal truncation considering n_m modes, as described in eq. (2). Indeed, the FRF is approximated as:

$$\boldsymbol{H}(\omega,\boldsymbol{\xi}) \approx \tilde{\boldsymbol{H}}(\omega,\boldsymbol{\xi}) = \sum_{m=1}^{n_m}\frac{\tilde{\boldsymbol{\phi}}_m(\boldsymbol{\xi})\tilde{\boldsymbol{\phi}}_m^T(\boldsymbol{\xi})}{\tilde{\omega}_m(\boldsymbol{\xi})^2 - \omega^2 + i2d_m\omega\tilde{\omega}_m(\boldsymbol{\xi})}, \tag{8}$$

where the symbol $\tilde{(\cdot)}$ denotes a quantity which is being approximated. The practical implementation of eq. (8) demands calculating approximate mode shapes and natural frequencies, respectively. Such a task is performed by projecting the eigenvalue/eigenvector problem in eq. (3) into a reduced basis [5, 6]. The reduced basis consists of the first n_m mode shapes of the system under consideration evaluated at a certain reference value of the interval variables $\boldsymbol{\xi}_0$, which can be, e.g. the mid-point of the associated interval. This reduced basis is denoted in the following as $\boldsymbol{\Phi}(\boldsymbol{\xi}_0)$ and it is a matrix of dimension $n_d \times n_m$. Taking into account this basis, the approximate mode shapes are equal to:

$$\left[\tilde{\boldsymbol{\phi}}_1(\boldsymbol{\xi}) \ \ldots \ \tilde{\boldsymbol{\phi}}_m(\boldsymbol{\xi})\right] = \boldsymbol{\Phi}(\boldsymbol{\xi}_0)\boldsymbol{\Psi}(\boldsymbol{\xi}) \tag{9}$$

where $\boldsymbol{\Psi}(\boldsymbol{\xi})$ is an $n_m \times n_m$ matrix whose columns are the eigenvectors associated with the problem:

$$\left(\boldsymbol{\Phi}(\boldsymbol{\xi}_0)^T\boldsymbol{K}(\boldsymbol{\xi})\boldsymbol{\Phi}(\boldsymbol{\xi}_0) - \tilde{\omega}_m^2(\boldsymbol{\xi})\boldsymbol{\Phi}(\boldsymbol{\xi}_0)^T\boldsymbol{M}(\boldsymbol{\xi})\boldsymbol{\Phi}(\boldsymbol{\xi}_0)\right)\boldsymbol{\psi}_m(\boldsymbol{\xi}) = 0, \quad m = 1,\ldots,n_m. \tag{10}$$

Note that the reduced eigenvalue/eigenvector problem in eq. (3) provides the approximate values of the natural frequencies $\tilde{\omega}_m(\boldsymbol{\xi})$. Moreover, as it is expected that $n_m \ll n_d$, the eigenvalue/eigenvector problem in eq. (10) is numerically less demanding than its counterpart in eq. (3).

With the approximate FRF obtained from eq. (8), one can solve the optimization problem of eqs. 6 and 7 by replacing $H_{i,j}(\omega, \boldsymbol{\xi})$ by $\tilde{H}_{i,j}(\omega, \boldsymbol{\xi})$. Given the potential discrepancies between an approximate model and the exact solution, a recommended approach is to recompute the approximate response bounds by evaluating the exact model only twice. These exact analyses rely on the optimal values of the uncertain input parameters ($\boldsymbol{\xi}^*$) identified for each bound by the reduced-order model. As a result, three exact analyses are required for the ROM: one to determine the optimal combination of uncertain parameters for both bounds and two (one for each bound) to accurately estimate the response bounds.

5 Example

To illustrate the implementation of the proposed strategy for interval dynamic analysis, the following test example is analyzed, adapted from [16]. The example corresponds to a truss structure constructed of steel bars, as illustrated in Figure 1. The truss geometry can be described as a superposition of identical tetrahedra, and is represented using three-dimensional bar elements. The FE model comprises 74 nodes, 210 degrees-of-freedom (DOF), and 288 bar elements. Both the geometry and the properties of the system are considered deterministic. Each bar is assigned a specific weight of 78 [kN/m^3] and a cross-section area of 4×10^{-3} [m^2]. It is assumed that the structure exhibits classical modal damping, with a damping ratio of 0.02. The Young's modulus of the bars is equal to 2×10^{11} [N/m^2]. The nodes located at coordinates (0,0,0) [m], (6,0,0) [m], (0,20,0) [m] and (6,20,0) [m] are simply supported. At positions (3,15,-2) [m] and (2,6,0) [m] there are additional masses, m_1 and m_2, respectively, of uncertain weight placed in the truss structure (see highlighted red nodes in Figure 1). Each of these masses is characterized by the interval $[5 \times 10^3, 10 \times 10^3]$ [kg].

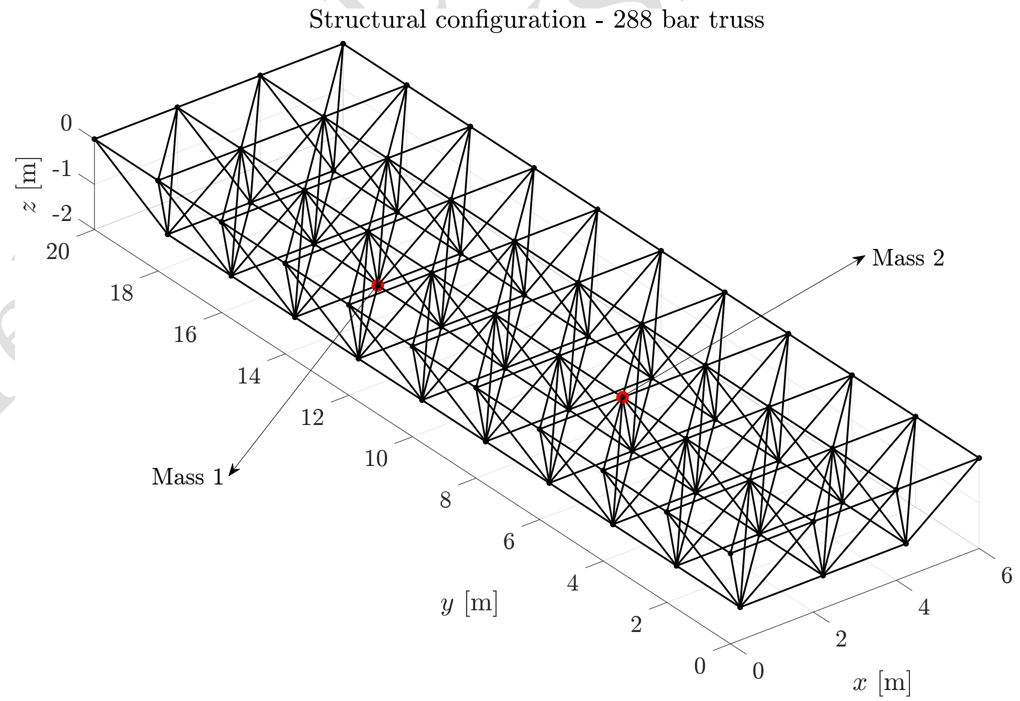

Figure 1: Schematic of the truss structure. Nodes highlighted in red represent the positions of the uncertain masses.

It is of interest to estimate the uncertainty in the maximum frequency response function (FRF), associated with the vertical DOF at the position of the first mass m_1 (denoted as V_{m_1}), due to the action of a vertical unitary load at the position on the second mass m_2 (denoted as V_{m_2}). Therefore, the objective of this study is to apply a reduced-order model (ROM) to identify the variation of the maximum FRF $|H_{V_{m_1},V_{m_2}}(\omega, \boldsymbol{\xi})|$. The ROM includes $n_m = 10$ modes, which are retained for dynamic analysis. The reference value of the interval variables $\boldsymbol{\xi}_0$ for the construction of the ROM (see eq. 10) is assumed to be equal to the mid-point of the interval quantities for the uncertain masses, i.e., $\boldsymbol{\xi}_0 = [7.5 \times 10^3, 7.5 \times 10^3]$ [kg]. To estimate the bounds of the response of interest, the Particle Swarm Optimization scheme [17] is applied.

Since only two uncertain parameters are considered (i.e., two uncertain masses), the behavior of the maximum FRF within the search space can be visualized. Figure 2 shows the exact (left) and approximate (right) response considering the ROM. In other words, the right part represents the reduced-order model constructed by using the mid-point of the interval quantities for the uncertain masses, i.e., $\boldsymbol{\xi}_0 = [7.5 \times 10^3, 7.5 \times 10^3]$ [kg]. It is observed that although the two solutions are not identical, the reduced-order model can capture the behavior of the FRF, particularly for the extreme values.

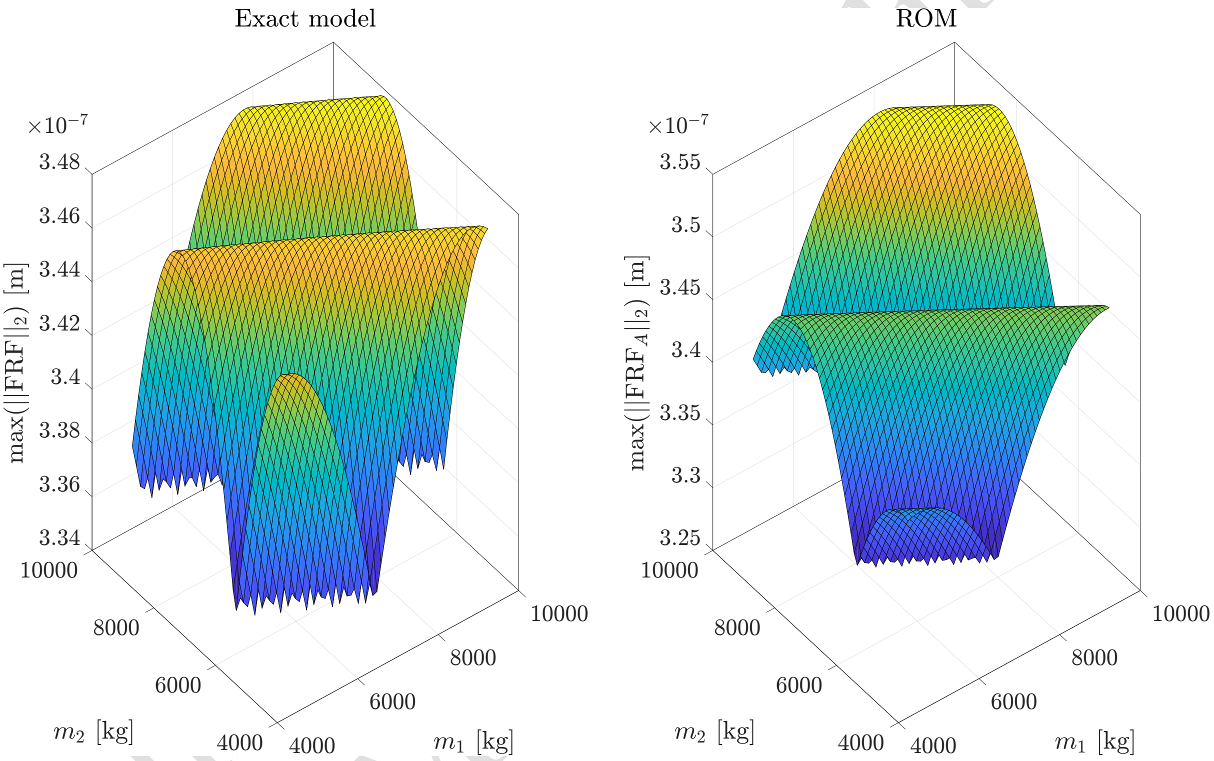

Figure 2: Maximum FRF trust structure exact model and ROM over the search space.

Tables 1 and 2 show the results for optimizing the bounds of the maximum FRF considering the exact model, only the ROM, and the ROM with the additional exact analyses for the bounds. As expected, the application of only the reduced-order model is not sufficient to accurately approximate the response of interest. An improvement in the extreme values of the response is observed with the inclusion of two additional analyses, one for each limit. The error of the proposed method for the lower bound is 2.0417×10^{-4} while for the upper bound is 7.0942×10^{-4}. Note how these results are obtained with a noticeable decrease in the number of exact evaluations of the system. Thus, these results highlight the advantage of using a ROM to compute the maximum FRF of a linear dynamical system.

Table 1: Lower bound maximum FRF trust structure.

Method	m_1 [kg]	m_2 [kg]	$\max(\lvert H_{V_{m_1},V_{m_2}}(\omega, \boldsymbol{\xi})\rvert)$ [m]	No. exact analysis
Exact model	5×10^3	6.6561×10^3	3.3425×10^{-7}	5322
ROM	5×10^3	6.6616×10^3	3.2948×10^{-7}	1
ROM + exact analysis	5×10^3	6.6616×10^3	3.3432×10^{-7}	3

Table 2: Upper bound maximum FRF trust structure.

Method	m_1 [kg]	m_2 [kg]	$\max(\lvert H_{V_{m_1},V_{m_2}}(\omega,\boldsymbol{\xi})\rvert)$ [m]	No. exact analysis
Exact model	10×10^3	8.3099×10^3	3.4774×10^{-7}	3403
ROM	10×10^3	8.5708×10^3	3.5295×10^{-7}	1
ROM + exact analysis	10×10^3	8.5708×10^3	3.4749×10^{-7}	3

6 Summary and conclusions

This paper presents an approach for performing interval analysis of linear dynamical systems. The study focused on estimating the bounds of the maximum FRF of a truss structure, due to the addition of two uncertain masses.

According to the results, the proposed method, which utilizes a reduced-order model approach, significantly reduces the computational cost associated with uncertainty quantification in an interval context. Only three exact analyses are required to estimate the response extremes accurately.

Nevertheless, the presented results should be regarded as a first approach to ROM analysis for dynamical systems under epistemic uncertainty. Work remains to be done to investigate more appropriate ways to construct the reduced basis in terms of error monitoring of the reduced-order model. Future work will also explore the application of this method to more complex systems and investigate its potential for other types of uncertainty description techniques (e.g., interval fields).

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