Isogeometric Analysis for Coping with Geometric Uncertainty in Mechanical Systems

Nataly A. Manque[1], Jan Liedmann[2], Franz-Joseph Barthold[2], Marcos A. Valdebenito[1] and Matthias G.R. Faes[1]
[1] *Chair for Reliability Engineering, TU Dortmund University, Dortmund 44227, Germany*

[2] *Institut für Baumechanik, Statik und Dynamik, TU Dortmund University, Dortmund 44227, Germany*

Abstract. Geometric uncertainty poses a significant challenge for industrial manufacturing. This uncertainty, coupled with geometric complexity, can lead to significant discrepancies between numerical simulations and the actual behavior of systems. Typically, the development of robust numerical models is limited due to epistemic uncertainty in the initial design stage. In these cases, interval methods can quantify these uncertainties. However, interval methods are computationally expensive, especially when finite elements (FE) are used to describe the system. This computational cost arises from the need to iteratively explore different geometries, requiring repeated FE analysis. Given this challenge, this paper explores the potential of Isogeometric Analysis (IGA) for quantifying geometric uncertainties characterized by intervals. IGA provides a means to approximate solution fields in numerical analysis using Non-Uniform Rational B-Splines (NURBS). The main advantage of IGA is its ability to manipulate the geometry, which facilitates the calculation of sensitivities. A variational formulation that allows the simultaneous computation of structural response and sensitivities is applied. This sensitivity information is used for gradient-based optimization in the interval analysis. The proposed strategy is tested for estimating uncertain stress triaxiality in a linear hook system.

Keywords: Isogeometric analysis (IGA); Uncertainty quantification; Geometric uncertainty; Interval analysis; Stress triaxiality.

1. Introduction

Geometrical uncertainties are inherent in manufacturing processes and arise from a variety of sources, including tool wear, machine imperfections, and material deformation (Cheng et al., 2023). These uncertainties can significantly affect the performance and reliability of mechanical systems, resulting in operational inefficiencies and variations in output quality. Managing these uncertainties is critical to ensuring the robustness and durability of mechanical components. Interval analysis techniques have emerged as a valuable tool for quantifying epistemic uncertainty (Faes and Moens, 2020), allowing engineers to characterize the range of potential deviations in system parameters and translate this information into the variation of the response of interest. Nevertheless, traditional interval analysis approaches, especially when applied to finite elements (FE), can be computationally demanding, especially for complex systems when the uncertainty is geometry-related.

One promising approach to address this numerical cost is Isogeometric Analysis (IGA) (Hughes et al., 2005). IGA allows solution fields to be computed by directly leveraging the geometric representations used in Computer-Aided Design (CAD) systems. By using Non-Uniform Rational B-Splines (NURBS) to describe geometries, IGA offers several advantages over traditional finite element methods, including higher order continuity of the basis functions and improved geometric accuracy (Agrawal and Gautam, 2018). These features make IGA particularly well suited for handling geometric uncertainty, as it can provide more faithful representations of complex shapes and structures. In addition, the inherent flexibility of IGA allows for efficient manipulation of geometry without the need for costly remeshing procedures, potentially reducing the computational overhead of uncertainty quantification tasks, especially when dealing with interval analysis.

In this context, this paper presents an approach that combines the advantages of IGA with interval analysis to efficiently quantify geometric uncertainties in mechanical systems. For the propagation of geometric uncertainties, the application of a Gradient-based Optimization (GBO) algorithm is proposed to determine both the lower and upper bounds of the system response. From the sensitivities of the IGA model, the gradient of the objective function is calculated concerning each geometric uncertain parameter. Exploiting the key advantage of IGA to manipulate geometry, a variational formulation is applied that allows the simultaneous computation of structural response and sensitivities (Liedmann and Barthold, 2020). The proposed strategy is tested for estimating uncertain stress triaxiality in a linear 2D hook system with uncertain radius and thickness.

2. Formulation of the problem

2.1. Governing equations

The focus of this work is the study of linear systems subject to static loads. Consider that the geometric parameters of the system are difficult to determine precisely due to uncertainties in the manufacturing process. As a result, the geometric input parameters are subject to epistemic uncertainty and are represented as a vector $\boldsymbol{x}$ of size n_x. To perform structural design calculations for this system, it is necessary to solve a set of partial differential equations (PDEs). The numerical model $\mathcal{M}(\boldsymbol{x})$ typically provides an approximate solution to these PDEs. It is important to note that the model $\mathcal{M}(\boldsymbol{x})$ depends on the uncertain geometric parameters $\boldsymbol{x}$. In addition, by applying numerical methods, the model yields a response $\boldsymbol{y}$, which is defined as

$$\mathcal{M}(\boldsymbol{x}) : \boldsymbol{y} = m(\boldsymbol{x}), \tag{1}$$

where m is a response function operator that maps the geometric uncertain input parameters $\boldsymbol{x}$ to the output response $\boldsymbol{y}$. This response can include various quantities of interest, such as displacements, stresses, or strain fields. In this work, the response of interest corresponds to stress triaxiality, which is defined in the following section. It should be noted that during the mapping with m, the behavior of the system, given by its response $\boldsymbol{y}$, is influenced by uncertain geometric variables $\boldsymbol{x}$. Therefore, the response of the system is also subject to uncertainties.

The construction of the numerical model $\mathcal{M}(\boldsymbol{x})$ using the traditional finite element method can be very time-consuming, especially if the uncertainty is related to the geometry. Consequently, the exploration of alternative methods becomes essential to reduce computational costs and increase efficiency when analyzing systems with complex geometries and uncertain parameters. In this paper, Isogeometric Analysis (IGA) is investigated as an alternative method due to its advantages in handling geometry.

2.2. Stress triaxiality

In manufacturing design, users are interested in the study of damage states (Tekkaya et al., 2020), as well as the initiation of fracture processes (Boeddecker et al., 2023). To achieve this, the analysis of stresses resulting from numerical simulations $\mathcal{M}(\boldsymbol{x})$ is crucial. Stress triaxiality is one of the most important factors in controlling such problems. The stress triaxiality index provides valuable information about the material behavior under complex loading conditions. Stress triaxiality $\sigma_{ST}(\boldsymbol{x})$ is, by definition, the ratio of the hydrostatic stress $\sigma_M(\boldsymbol{x})$ to a deformation-related deviatoric stress contribution $\sigma_V(\boldsymbol{x})$

$$\sigma_{ST}(\boldsymbol{x}) = \frac{\sigma_M(\boldsymbol{x})}{\sigma_V(\boldsymbol{x})}, \tag{2}$$

where, for general plane stress conditions, the hydrostatic stress corresponds to

$$\sigma_M(\boldsymbol{x}) = \frac{\sigma_{11}(\boldsymbol{x}) + \sigma_{22}(\boldsymbol{x})}{2}, \tag{3}$$

where σ_{11} and σ_{22} are the principal stresses, and the deviatoric stress contribution can be considered as the equivalent von Misses stress

$$\sigma_V(\boldsymbol{x}) = \sqrt{\sigma_{11}^2(\boldsymbol{x}) + \sigma_{22}^2(\boldsymbol{x}) - \sigma_{11}(\boldsymbol{x})\sigma_{22}(\boldsymbol{x}) + 3\sigma_{12}^2(\boldsymbol{x})}, \tag{4}$$

where $\sigma_{12}(\boldsymbol{x})$ is the shear stress.

After identifying the response of the system ($\sigma_{ST}(\boldsymbol{x})$ for this work) and the geometric uncertain parameters, the next step is to characterize the uncertainty in these parameters. Several techniques are available to characterize the uncertainty that affects stress triaxiality. One approach is to use interval analysis following a set-based method. The essential definitions for incorporating this uncertainty using interval analysis are discussed in the next section.

3. Interval analysis

The available data on element thicknesses, lengths, and shapes can be highly affected by epistemic uncertainty at an early design stage. In most cases, this data is not sufficient to build a robust numerical model for predicting the behavior of mechanical components. A means of representing this type of uncertainty is to resort to interval analysis (Moore, 1979). An interval or interval scalar is a convex subset of the domain of real numbers $\mathbb{R}$. An interval parameter x^I is defined by

$$x^I = [\underline{x}, \bar{x}] = \{x \in \mathbb{R} \mid \underline{x} \leq x \leq \bar{x}\}, \tag{5}$$

where $\underline{x}$ represents the lower bound and $\bar{x}$ corresponds to the upper bound of x^I. Therefore, x^I contains all possible values that an uncertain input parameter can take, with no assumption made regarding the likelihood of those values (Sofi and Romeo, 2016).

In most cases, there is more than one uncertain parameter. In this situation, the definition of an interval vector is useful. An interval vector $\boldsymbol{x}^I$ is a vector in which each element is an interval

$$\boldsymbol{x}^I = \begin{Bmatrix} x_1^I \\ x_2^I \\ \vdots \\ x_a^I \end{Bmatrix} = \left\{ \boldsymbol{x} \in \mathbb{R}^a \mid x_i \in x_i^I \right\}, \tag{6}$$

with $\boldsymbol{x}^I \in \mathbb{IR}^a$, the domain of closed real-valued interval vectors of size a. Consequently, an a-dimensional interval vector describes a hypercube in a-dimensional space. The lower and upper bounds of the interval scalar entries in the interval vector $\boldsymbol{x}^I$ determine the vertices of this hypercube (Faes and Moens, 2020). Thus, the fundamental idea of interval analysis is to search, from a hypercube $\boldsymbol{x}^I$, for those parameter realizations that yield the extreme response of the system. In the case that the response of interest is scalar y^I, e.g. stress triaxiality, the optimization problem corresponds to

$$\underline{y} = \min_{\boldsymbol{x} \in \boldsymbol{x}^I} m(\boldsymbol{x}) \tag{7}$$

$$\bar{y} = \max_{\boldsymbol{x} \in \boldsymbol{x}^I} m(\boldsymbol{x}), \tag{8}$$

where $y^I = [\underline{y}, \bar{y}]$ is the interval response of the system which is defined by its lower $\underline{y}$ and upper $\bar{y}$ bounds.

Typically y^I is calculated following a global optimization approach. Nevertheless, the use of a global optimization approach to compute the response bounds requires repeated deterministic analyses by examining different realizations of the uncertain geometric input parameters. Needless to say, the numerical cost associated with finding both response bounds is directly influenced by the nature of $m(\boldsymbol{x})$ and hence the response (Moens and Hanss, 2011). Thus, the method used to construct $\mathcal{M}(\boldsymbol{x})$ has a strong influence on the numerical cost of finding the response of interest. In particular, when using the finite element method and considering that the uncertainty lies in the geometry, it would be necessary to modify the discrete representation of the system (i.e., the mesh) for each of the realizations needed to find the response bounds during the optimization stage. This drawback is due to the decoupling between the meshing procedure and the numerical computation of the field responses. This difficulty can be overcome by using a technique that allows the treatment of both geometry and solution domains at the same time. In the following section, Isogeometric Analysis is presented as a viable alternative for the propagation of geometric uncertainty.

4. Isogeometric analysis

Isogeometric analysis (IGA) has been proposed as an alternative to finite element analysis (FEA) for solving partial differential equations in various engineering disciplines (Hughes et al., 2005). The two methods are based on the same fundamental concepts, but unlike FEA, in IGA the geometry of the analyzed structure is not approximated by polynomial shape functions (e.g., Lagrangian basis functions), but rather by a smooth geometry description used in Computer-Aided Design (CAD). These descriptions are usually based on Non-Uniform Rational B-Splines (NURBS). NURBS curves, surfaces, and volumes can be defined by knot vectors $\mathbf{\Xi}$ and control points. The knot vectors must have $n+p+1$ increasing entries, called knots ξ_i, of the form

$$\mathbf{\Xi} = \{\xi_1, \xi_2, \ldots, \xi_{n+p+1}\}\,, \tag{9}$$

and define the parametric space as well as the NURBS order p. It also defines the C^{p-1-k} continuity conditions at the knots, where k denotes the number of repetitions of a specific knot in the knot vector $\mathbf{\Xi}$. Further, n is the total number of NURBS basis functions that are defined by,

$$R_{i,p}(\xi) = \frac{w_i N_{i,p}(\xi)}{W(\xi)}, \quad 1 \leq i \leq p+1, \quad \text{with} \quad W(\xi) = \sum_{i=1}^{\mathsf{n_{cp}}} w_i N_{i,p}(\xi), \tag{10}$$

where $\mathsf{n_{cp}}$ is the total number of NURBS control points, $w_i > 0$ are weight factors and $N_{i,p}$ are B-spline basis functions of order p (Agrawal and Gautam, 2018). NURBS curves $\mathbf{C}(\xi)$ and surfaces $\mathbf{S}(\xi,\eta)$ are respectively described by,

$$\mathbf{C}(\xi) = \sum_{i=1}^{n} R_{i,p}(\xi)\mathbf{P}_i, \quad \mathbf{S}(\xi,\eta) = \sum_{i=1}^{n}\sum_{j=1}^{m} R_{i,p}(\xi)R_{j,q}(\eta)\mathbf{P}_{i,j}, \tag{11}$$

where $\mathbf{P}$ stores the control point coordinates, and m and q correspond to the number of NURBS basis functions, and the NURBS order in the second space dimension, respectively. Note that η represents a second parametric dimension.

Following the finite element formulation, the key point to define the IGA formulation is to resort to the weak equilibrium equation. The discretized matrix form of the weak equilibrium equation differs from the FEA formulation only in the choice of shape functions, which will correspond to the NURBS defined in Eq. 10. For a detailed explanation of the procedure, the reader is referred to (Hughes et al., 2005). It is noteworthy that, unlike FEA, in IGA the response in displacements is given in the positions of the control points. With the solution of the IGA model for the displacement, any response function of interest can be computed within a post-processing step similar to FEA.

5. Sensitivity analysis

Design sensitivity analysis helps quantify the change of any response function $f(\boldsymbol{y}(\boldsymbol{x}),\boldsymbol{x})$ concerning alterations in chosen design parameters $\boldsymbol{x}$. By employing variational sensitivity analysis, as discussed in e.g. (Barthold et al., 2016; Liedmann and Barthold, 2020), this change can be expressed as

$$\delta f = \delta_y f + \delta_x f = \left[\frac{\partial f}{\partial \boldsymbol{y}}\right] \delta \boldsymbol{y} + \left[\frac{\partial f}{\partial \boldsymbol{x}}\right] \delta \boldsymbol{x}. \tag{12}$$

Therefore, the discrete sensitivity relation of the stress triaxiality (see Eq. 2) can be expressed by

$$\delta\sigma_{ST} = \left[\frac{\partial \sigma_{ST}}{\partial \sigma_M}\frac{\partial \sigma_M}{\partial \boldsymbol{\sigma}} + \frac{\partial \sigma_{ST}}{\partial \sigma_V}\frac{\partial \sigma_V}{\partial \boldsymbol{\sigma}}\right] \delta\boldsymbol{\sigma}, \tag{13}$$

with $\boldsymbol{\sigma}$ the stress solution vector from the IGA model.

The sensitivities of Eq. 13 can be used to solve the optimization problem of Eq. 7 and Eq. 8, following a gradient-based optimization approach.

6. Example

The proposed method is applied to estimate the maximum stress triaxiality of a linear two-dimensional steel hook system. In the analysis, plane stress conditions are assumed. The bottom end of the hook is fixed and a load of 20 [kN] is applied to the top end. The material properties of the hook system are assumed to be deterministic and equal to $E = 2 \times 10^5$ [N/mm^2] for Young's modulus and $\nu = 0.3$ for Poisson's ratio. For the system geometry, it is assumed that the radius and thickness values are uncertain due to a lack of knowledge in the early design stage. These geometric quantities are characterized by the intervals $r^I = [10, 50]$ [mm] and $t^I = [15, 40]$ [mm] for the radius and thickness, respectively. Figure 1 shows the IGA model for stress triaxiality analysis. Note that in this figure, the geometry representation is schematized considering the midpoints of the intervals, that is, $\mu_{r^I} = 30$ [mm] and $\mu_{t^I} = 27.50$ [mm].

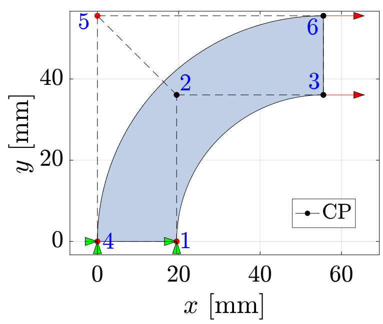

Figure 1. Hook 2D model for stress triaxiality analysis. The geometry considered corresponds to that described by the midpoints of the intervals associated with the radius and thickness.

The NURBS surface used to represent the hook system is constructed based on $\mathsf{n_{cp}} = 6$ control points. For the definition of the NURBS surface, quadratic elements with overlapping are considered. The polynomial degree p of the splines associated with the knot vector in the $x-$direction is two, while in the $y-$direction is one. On the other hand, the multiplicity of the knots k is one and zero for the $x-$direction and the $y-$direction, respectively. For both directions, the weights $w = [1, \frac{1}{\sqrt{2}}, 1]$

are associated with the control points of the inner and outer curves that allow to represent the hook geometry.

A gradient optimization approach is used to determine the lower and upper bounds of the maximum stress triaxiality σ_{ST} in the hook system. The starting point for the optimization scheme has been considered as the midpoint of the intervals. The results were compared by considering the Vertex Method (VM) (Dong and Shah, 1987) and Particle Swarm optimization scheme (see Table I). The results for the lower bound of the maximum stress triaxiality of the hook system are shown in Table I. It can be seen that all evaluated methods identify the lower bound of the maximum stress triaxiality as 0.4420 for a radius equal to $r = 10$ [mm] and a thickness of $t = 40$ [mm]. Nevertheless, the Gradient-based Optimization (GBO) method appears to be the most efficient after the Vertex Method (VM), requiring only five deterministic analyses of the hook system to identify this lower bound, emphasizing the numerical advantage of the proposed strategy. Note that although the VM gives the exact results for the lower bound, this method is accurate only if the response is monotonous over the search space, which is not the case for the stress triaxiality response considered in this study (see Figure 2).

Table I. Results of optimization maximum stress triaxiality Hook 2D lower bound.

Method	r [mm]	t [mm]	$\max(\sigma_{ST})$	No. Analysis
Vertex Method (VM)	10	40	0.4420	4
Particle Swarm Optimization (PSO)	10	40	0.4420	2254
Gradient-based Optimization (GBO)	10	40	0.4420	5

Since only two uncertain parameters are considered, the behavior of the maximum stress triaxiality within the search space can be visualized. Note that the information about the sensitivity of the response facilitates the fast convergence of the algorithm to the optimal value. This is an indication of the efficiency and effectiveness of the GBO approach coupled with the IGA model in the context of interval analysis. Figure 3 shows the resulting geometry for the hook system with the optimum values of radius and thickness for the lower bound of the response. As expected, the lower bound of maximum stress triaxiality is associated with a thicker hook geometry. It should be noted that the high values of the stress triaxiality are located in the outer curve of the hook. These regions of higher stress triaxiality (closer to 0.4) are likely to be more susceptible to failure under load because they indicate high-stress concentration.

The results of the optimization procedure for the maximum stress triaxiality upper bound are shown in Table II. For this bound, due to the non-monotonicity of the response, the Vertex Method underestimates the optimum. Note that all optimization methods used to find the upper bound of $\max(\sigma_{ST})$, obtain the same optimum value of maximum stress triaxiality by different radius and thickness combinations. This is caused by the flat behavior of stress triaxiality over the search space observed in Figure 4. As with the lower bound of the response, the GBO method appears to be the most efficient, requiring only eight deterministic analyses of the system.

The resulting geometry for the hook system with the optimum values of radius and thickness for the upper bound of the response is shown in Figure 5. A thinner hook geometry is associated with

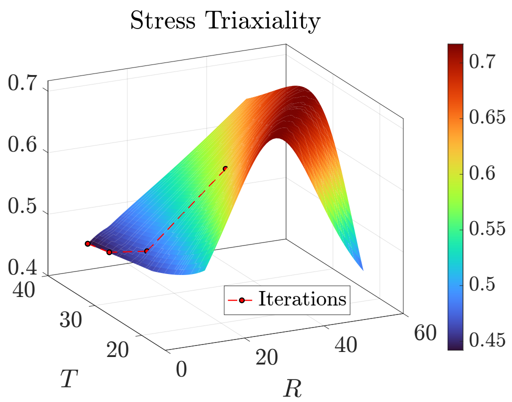

Figure 2. Distribution of the maximum stress triaxiality over the search space and iterations performed for the GBO algorithm to find the lower bound.

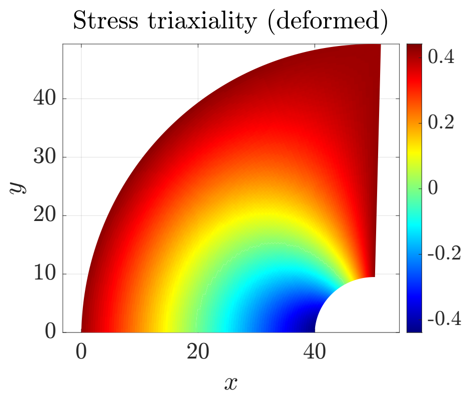

Figure 3. Resultant geometry and stress triaxiality for the lower bound results in the deformed shape. Axes in [mm].

the upper bound of the maximum stress triaxiality. Note that, as observed for the lower bound results, the maximum values of stress triaxiality are located in the outer curve of the hook. Again, these areas of higher stress triaxiality (closer to 0.7) are likely to be more susceptible to failure under load. Unlike the resulting geometry for the lower boundary, a wider range of stress triaxiality values is now observed in the hook shape.

Table II. Results of optimization maximum stress triaxiality Hook 2D upper bound.

Method	r [mm]	t [mm]	$\max(\sigma_{ST})$	No. Analysis
Vertex Method (VM)	50	40	0.6380	4
Particle Swarm Optimization (PSO)	49.6141	26.7778	0.7170	3803
Gradient-based Optimization (GBO)	36.0834	19.4750	0.7170	8

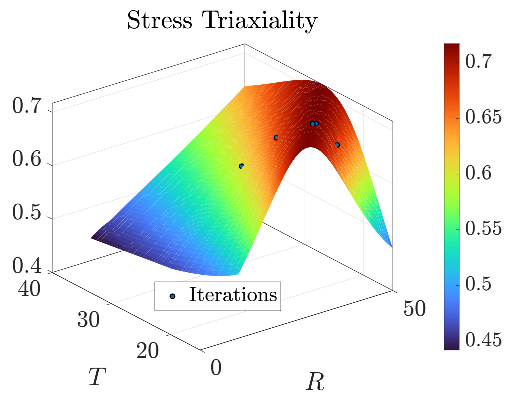

Figure 4. Distribution of the maximum stress triaxiality over the search space and iterations performed for the GBO algorithm to find the upper bound.

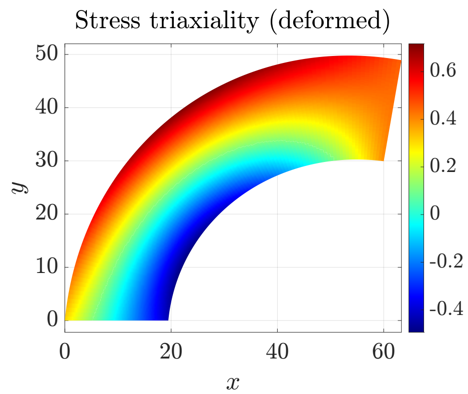

Figure 5. Resultant geometry and stress triaxiality for the upper bound results in the deformed shape. Axes in [mm].

7. Summary and conclusions

This paper presented a novel approach that explores the combination of Isogeometric Analysis (IGA) with interval analysis to efficiently quantify geometric uncertainties in mechanical systems. The study focused on estimating the bounds of maximum stress triaxiality in a 2D hook system with uncertain radius and thickness parameters.

According to the results, the proposed method, which utilizes the gradient-based optimization (GBO) approach, significantly reduces the computational cost associated with uncertainty quantification in an interval context. The efficiency of the method is due to the ability of the IGA model to directly manipulate geometry and compute sensitivities without the need for costly remeshing procedures, a common problem in traditional finite element methods (FEM).

Future work will explore the application of this method to more complex systems and investigate its potential for other types of uncertainty description techniques (e.g., interval fields).

Acknowledgements

Financial support by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) – Project-ID 278868966 – TRR188 is gratefully acknowledged.

References

Agrawal, V. and Gautam, S. S. IGA: A simplified introduction and implementation details for finite element users *Journal of The Institution of Engineers (India)*, Series C 100 (3): 561–585, 2018.

Barthold, F. -J., and Gerzen, N. and Kijanski, W. and Materna, D.
Efficient Variational Design Sensitivity Analysis, *Vol. 40 of Computational Methods in Applied Sciences, Springer International Publishing, Switzerland*, 2016.

Böddecker, M. and Faes, M. and Menzel, A. and Valdebenito, M. Effect of uncertainty of material parameters on stress triaxiality and lode angle in finite elasto-plasticity—a variance-based global sensitivity analysis, *Advances in Industrial and Manufacturing Engineering* 7, 100128, 2023.

Cheng, T. and Xiang, S. and Zhang, H. and Yang, J. New machining test for identifying geometric and thermal errors of rotary axes for five-axis machine tools. *Measurement*, 223, 2023.

Dong, W. and Shah, H. C. Vertex method for computing functions of fuzzy variables, *Fuzzy Sets and Systems* 24 (1): 65–78, 1987.

Faes, M. and Moens, D. Recent trends in the modeling and quantification of non-probabilistic uncertainty. *Archives of Computational Methods in Engineering*, 27: 633–671, 2020.

Hughes, T. J. R. and Cottrell, J. A. and Bazilevs, Y. Isogeometric analysis: CAD, finite elements, NURBS, exact geometry and mesh refinement. *Computer Methods in Applied Mechanics and Engineering*, 194: 4135–4195, 2005.

Liedmann, J. and Barthold, F. -J. Variational sensitivity analysis of elastoplastic structures applied to optimal shape of specimens. *Structural and Multidisciplinary Optimization*, 61: 2237–2251, 2020.

Moens, D. and Hanss, M. Non-probabilistic finite element analysis for parametric uncertainty treatment in applied mechanics: Recent advances, *Finite Elements in Analysis and Design* 47 (1): 4–16, 2011.

Moore, R. E. Methods and Applications of Interval Analysis, *Society for Industrial and Applied Mathematics*, 1979.

Sofi, A. and Romeo, E. A novel Interval Finite Element Method based on the improved interval analysis, *Computer Methods in Applied Mechanics and Engineering*, 311: 671–697, 2016.

Tekkaya, A. and Bouchard, P.-O. and Bruschi, S. amd Tasan, C. Damage in metal forming, *CIRP Annals* 69 (2):600–623, 2020.