Bayesian updating of conditional failure probability using method of moments

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Abstract

Bayesian updating reduces epistemic uncertainty for more reliable predictions, but characterizing the distribution of conditional failure probability with measurement data is complex. This study proposes an efficient and accurate method to fully describe the probabilistic characteristics of the updated conditional failure probability. It formulates the first three raw moments of the updated conditional reliability index and uses weighted sparse grid numerical integration to evaluate these moments. A shifted lognormal distribution is then used to approximate the probability density function of the updated conditional reliability index, allowing for the determination of the mean, quantiles, and distribution of the updated conditional failure probability with information reuse. An illustrative example was conducted to demonstrate the method's performance, with results compared against benchmarks from MCMC combined with MCS.

1 Introduction

Uncertainties are intrinsic to nearly all engineering problems, arising from incomplete knowledge, experimental data limitations, modeling assumptions, measurement errors, and environmental effects [1]. These uncertainties can generally be classified into two types: aleatory uncertainty and epistemic uncertainty [2]. To address these uncertainties rationally, many scholars have developed and applied probabilistic frameworks for their quantification [3-5]. Aleatory uncertainty is typically represented by a probability density function (PDF) or a cumulative distribution function (CDF) [3]. In contrast, epistemic uncertainty, which results from a lack of knowledge, can be reduced with additional information. [6]

Recent advancements in monitoring and sensor technology have facilitated the collection of relevant data from inspections and measurements, helping to reduce epistemic uncertainties [7]. Bayesian inference is a suitable method for incorporating such data into probabilistic models, ensuring consistent treatment of uncertainty and improving the reliability of failure probability analysis [8]. Bayesian theory allows for the updating of epistemic uncertainty with new data, leading to a "posterior" failure probability that reflects the impact of new information on the structural model [9-11]. This enhances the accuracy of probabilistic estimates for structural failure, thereby increasing confidence in future predictions and enabling more effective risk-based decision-making.

The evaluation of the expected value of posterior failure probability, often referred to as "robust posterior failure probability," has been an active research area. For instance, Der Kiureghian [12] used Bayesian methods to combine data measurements with reliability analysis, deriving posterior distributions of uncertain parameters through conjugate prior distributions and obtaining the expected value of posterior failure probability. Beck and Au [10] proposed an adaptive Markov Chain Monte Carlo (MCMC) method to evaluate this expected value based on the Metropolis-Hastings algorithm. Papadimitriou et al. [13] presented a framework using the theorem of total probability to update robust failure probability. Jensen et al. [14] developed a method using transitional MCMC simulation to update uncertain parameters and calculate robust failure probability. Fan et al. [15] extended the probability density evolution method for deteriorating structures. Li et al. [16] proposed an efficient method to directly assess the expected value of posterior failure probability, avoiding iterative reliability analyses.

However, focusing solely on the expected value of failure probability, which encapsulates both aleatory and epistemic uncertainty, may not be sufficient for efficient and unbiased decision-making. To fully characterize posterior failure probability, methods like Monte Carlo simulation (MCS) [17] are commonly used. These methods sample posterior failure probability based on updated uncertain parameters but can be computationally expensive and lack explicit expressions. To address these drawbacks, approximate methods like Laplace's asymptotic approximation [13] and first-order approximation approaches [2] have been introduced, though they may not always be broadly applicable or efficient for multiple updates.

This study proposes an efficient framework for fully characterizing posterior failure probability considering Bayesian updating. The method separates aleatory and epistemic uncertainties and uses a shifted lognormal distribution [18, 19] to model the posterior reliability index. Weighted sparse grid algorithms [20] and information reuse [16] are employed to assess the statistical moments of the posterior reliability index, avoiding repeated reliability analyses and reassessments for each update.

The structure of this paper is as follows. Section 2 formulates the reliability updating problem. Section 3 proposes an efficient method for characterizing posterior failure probability based on Bayesian updating. Section 4 presents a numerical example to demonstrate the method's efficiency and effectiveness, using results from MCMC combined with MCS as benchmarks. Section 5 summarizes the main conclusions.

2 Problem Statement

Due to the influence of the external effect and the limitations of measurement, it would be extremely challenging to obtain precise information about the distribution parameters of random variables. Let the random vector $\mathbf{\Theta}$ describe the uncertain distribution parameters and its degree of imperfection is characterized by the distribution of $\mathbf{\Theta}$. In cases where uncertain distribution parameters are involved in the reliability analysis, the conditional probability of failure can be formulated as follows [12]:

$$P_f(\boldsymbol{\theta}) = \int_{G(\mathbf{X}|\mathbf{\Theta})\leq 0} f_{\mathbf{X}|\mathbf{\Theta}}(\mathbf{x},\boldsymbol{\theta})d\mathbf{x}\,, \tag{1}$$

where $P_f(\boldsymbol{\theta})$ denotes the conditional failure probability, $f_{\mathbf{X}|\mathbf{\Theta}}(\mathbf{x},\boldsymbol{\theta})$ denotes the conditional probability distribution function of $\mathbf{X}$ based on $\mathbf{\Theta}$, and $G(\mathbf{X}|\mathbf{\Theta})$ denotes the limit-state function in terms of $\mathbf{X}$ and $\mathbf{\Theta}$.

If the inspection data concerning the random variables associated with the uncertain distribution parameters can be obtained, it is possible to reduce the degree of uncertainty of these uncertain parameters. Based on the Bayesian theory [8], the distribution of these uncertain parameters can be modified by

$$f'_{\mathbf{\Theta}}(\boldsymbol{\theta}) = kL(\boldsymbol{\theta})\, f_{\mathbf{\Theta}}(\boldsymbol{\theta}) \tag{2}$$

where $f_{\mathbf{\Theta}}(\boldsymbol{\theta})$ represents the initial ("prior") distribution of uncertain parameters, which is usually derived from the existing information; $f'_{\mathbf{\Theta}}(\boldsymbol{\theta})$ represents the updated ("posterior") distribution of uncertain parameters, incorporating the newly acquired inspection data; $L(\boldsymbol{\theta})$ represents the likelihood function of the inspection data; and k represents a normalizing factor, which can be formulated as follows:

$$k = \int_{-\infty}^{\infty} L(\boldsymbol{\theta})\, f_{\mathbf{\Theta}}(\boldsymbol{\theta})\mathrm{d}\boldsymbol{\theta}\,. \tag{3}$$

When dealing with a sample of inspection data, represented as $\mathbf{x}_i$ where $i = 1, 2,\ldots, N_d$ from random variable vector $\mathbf{X}$, if it is possible to express the PDF of the inspection data $\mathbf{x}_i$ under $\boldsymbol{\Theta} = \boldsymbol{\theta}$ in an explicit form $f_{\mathbf{X}|\boldsymbol{\Theta}}(\mathbf{x}_i,\boldsymbol{\theta})$, then the likelihood function can be derived as

$$L(\boldsymbol{\theta}) = \prod_{i=1}^{N_d} f_{\mathbf{X}|\boldsymbol{\Theta}}(\mathbf{x}_i,\boldsymbol{\theta}) . \tag{4}$$

By replacing the initial samples of uncertain distribution parameters in Eq. (1) with the posterior samples obtained from Eq. (2), the updated conditional failure probability considering the inspection data can be obtained, which is given as

$$P_f^{up}(\boldsymbol{\theta}) = \int_{G(\mathbf{X}|\,\boldsymbol{\Theta})\le 0} f_{\mathbf{X}|\boldsymbol{\Theta}}(\mathbf{x},\boldsymbol{\theta}^{up})d\mathbf{x} , \tag{5}$$

where $P_f^{up}(\boldsymbol{\theta})$ is the updated conditional failure probability and $f_{\mathbf{X}|\boldsymbol{\Theta}}(\mathbf{x},\boldsymbol{\theta}^{up})$ is the conditional probability distribution function of $\mathbf{X}$ based on the updated uncertain distribution parameters $\boldsymbol{\Theta}$.

Subsequently, relying on the functional relationship between the failure probability and its associated reliability index, the updated conditional reliability index can be determined by

$$\beta^{up}\left(\boldsymbol{\Theta}\right) = \Phi^{-1}\left[1 - P_f^{up}(\boldsymbol{\Theta})\right], \tag{6}$$

where $\Phi^{-1}[\cdot]$ indicates the standard normal inverse cumulative density function.

As the uncertain distribution parameters are not deterministic, the updated conditional failure probability and the associated reliability index are inherently uncertain and can be assessed in a probabilistic manner. In general, the updated conditional failure probability and its associated reliability index can be treated as random variables, and therefore, they exhibit the same statistical characteristics as other random variables, including probability distribution functions and statistical moments.

In the context of structural risk analysis, providing a comprehensive depiction of the uncertainty associated with the updated conditional failure probability and its variations holds significant importance. In this regard, one should pay particular attention to three critical aspects: the mean value of the updated conditional failure probability, the quantiles of the updated conditional failure probability, and even the probability of the updated conditional failure probability.

To calculate the mean value of the updated conditional failure probability, it is recommended to employ some widely used techniques for evaluating the updated conditional failure probability under each set of given parameters $\boldsymbol{\Theta} = \boldsymbol{\theta}$. Consequently, the mean value of the updated conditional failure probability can be computed as

$$P_F^{up} = E\left[P_f^{up}(\boldsymbol{\theta})\right] = \int P_f^{up}(\boldsymbol{\theta}) f'_{\boldsymbol{\Theta}}(\boldsymbol{\theta})d\boldsymbol{\theta} . \tag{7}$$

To calculate Eq. (7), commonly used methods typically involve initially obtaining posterior samples of the uncertain distribution parameters, which are then utilized for the corresponding reliability analysis. Consequently, in the case of multiple updates, it becomes imperative not only to reacquire new posterior samples of uncertain distribution parameters but also to repeat the corresponding reliability analysis for each update. In addition, most existing studies do not provide a comprehensive description of the updated conditional failure probability, encompassing their quantiles and probability properties. Therefore, it is of great significance to establish a rational and efficient framework for fully describing the characteristics of the updated conditional failure probability.

3 Proposed method

To describe the updated conditional failure probability, one of the primary objectives is to approximate the distribution of the updated conditional failure probability. However, the support of the updated conditional failure probability may span several orders of magnitude, making it a challenging task to fit its distribution accurately. In contrast, the updated conditional reliability index exhibits a behavior which is easier to describe and has a strictly monotonic functional relationship with the updated conditional failure probability. Therefore, the primary focus of this study is to obtain the distribution function of the updated conditional reliability index based on the shifted lognormal distribution [18, 19], rather than to approximate the distribution function of the updated conditional failure probability. The essential concepts and steps are as follows: Firstly, calculate the first three moments of the updated conditional reliability index using the Smolyak-type quadrature formula. Subsequently, establish the distribution function of the updated conditional reliability index based on the shifted lognormal distribution [18, 19]. Finally, the mean, percentiles, and distribution of the updated conditional failure probability can be derived by utilizing the distribution of the updated conditional reliability index.

3.1 First four moments of the updated conditional reliability index

Based on the definition, the l-th raw moments of the updated conditional reliability index can be formulated as follows [21]:

$$\hbar_{l\beta}=E\left\{\left[\beta^{up}(\boldsymbol{\Theta})\right]^{l}\right\}=\int_{\boldsymbol{\theta}}\left[\beta(\boldsymbol{\Theta})\right]^{l} f'_{\boldsymbol{\Theta}}(\boldsymbol{\theta})d\boldsymbol{\theta}\,, \tag{8}$$

where $E[\cdot]$ denotes the expectation operator.

$$\hbar_{l\beta}=E\left\{\left[\beta(\boldsymbol{\Theta})\right]^{l}\xi(\boldsymbol{\theta})\right\}=\int_{\boldsymbol{\theta}}\left[\beta(\boldsymbol{\Theta})\right]^{l}\xi(\boldsymbol{\theta})\, f'_{\boldsymbol{\Theta}}(\boldsymbol{\theta})d\boldsymbol{\theta}\,, \tag{9}$$

where $\xi(\boldsymbol{\theta}) = kL(\boldsymbol{\theta})$ denotes the updating factor, signifying the part that varies with the updates.

From Eq. (9), it is can be observed that the l-th order raw moments of the updated conditional reliability index can also be computed as the mean value of $[\beta(\boldsymbol{\Theta})]^{l}\xi(\boldsymbol{\theta})$ under the initial PDF $f_{\boldsymbol{\Theta}}(\boldsymbol{\theta})$. Therefore, by integrating the integrand $[\beta(\boldsymbol{\Theta})]^{l}\,\xi(\boldsymbol{\theta})$ under the unchanged PDF $f_{\boldsymbol{\Theta}}(\boldsymbol{\theta})$, the l-th order raw moments of the updated conditional reliability index can be determined. To numerically solve Eq. (9), this study adopts the sparse grid numerical integration (SGNI) [20] due to its ability to strike a favorable balance between efficiency and accuracy when evaluating statistical moments. Furthermore, the points and associated weights can be readily determined once the accuracy level, denoted as q, has already been chosen. The fundamental concept behind SGNI is derived from the Smolyak algorithm [20], which employs a weighted linear combination of specialized tensor products to reduce the size of the integration grid [22]. Based on the SGNI, Eq. (9) can be derived as:

$$\hbar_{l\beta}=\sum_{\mathrm{i}\in H(q,N)}(-1)^{q+N-|i|}\binom{N-1}{q+N-|i|}\times\sum_{j_1=1}^{2i_1-1}\cdots\sum_{j_N=1}^{2i_N-1}\beta[T^{-1}(u_{j_1}),\cdots,T^{-1}(u_{j_N})]\xi[T^{-1}(u_{j_1}),\cdots,T^{-1}(u_{j_N})]p_{j_1}\cdots p_{j_N}\,, \tag{10}$$

where $T^{-1}(\cdot)$ represents the inverse normal transformation operator between the prior CDF of the uncertain parameters $F_{\boldsymbol{\Theta}}(\boldsymbol{\theta})$ the standard normal CDF $\Phi(\mathbf{u})$, in which $\mathbf{u}$ represents the standard normal random variable vector; $u_{j_h}=\sqrt{2}\xi_{j_h}$ and $p_{j_h}=\frac{1}{\sqrt{\pi}}\zeta_{j_h}$ (h = 1, …, N) represents the weights and estimation points within the standard normal space, which can be determined using the Gaussian-Hermite formula associated with the weight $\exp(-x^2)$ [23]; $i=(i_1,\cdots,i_N)\in\mathbb{N}_{+}^{N}$; $|i|=i_1+\cdots+i_N$ represents the sum of the multiple indices; the non-negative integer q represents the accuracy level, and the set $H(q, N)$ is expressed as

$$H(q,N)=\{\boldsymbol{i}\in\mathbb{N}_{+}^{N},\boldsymbol{i}\ge(1,\ldots,1):q+1\le\sum_{n=1}^{N}i_n\le q+N\}\,. \tag{11}$$

It is worth noting that once the specified accuracy level q is determined, the interpolation points of each uncertain parameter $\Theta_i = T^{-1}(u_i)$ (i =1 … N) are also determined and will not change under the unchanged prior distribution. This implies that when dealing with multiple updates, the values of the conditional reliability index for each set of interpolation points are determined and can be reused in subsequent updates.

After obtaining the first three raw moments of the updated conditional reliability index (i.e., $\hbar_{1\beta}$, $\hbar_{2\beta}$, and $\hbar_{3\beta}$), the first three moments of the updated conditional reliability index can be calculated as follows:

$$\mu_\beta = \hbar_{1\beta} , \tag{12}$$

$$\sigma_\beta = \sqrt{\hbar_{2\beta} - \hbar_{1\beta}^2} , \tag{13}$$

$$\alpha_{3\beta} = \left(\hbar_{3\beta} - 3\hbar_{2\beta}\hbar_{1\beta} + 2\hbar_{1\beta}^3\right) / \left(\sigma_\beta^3\right) . \tag{14}$$

3.2 Distribution of the updated conditional reliability index

Upon determining the first three central moments of the updated conditional reliability index, i.e., mean (μ_β), standard deviation (σ_β), and skewness ($\alpha_{3\beta}$), the PDF of the updated conditional reliability index can be estimated by using parameterized distributions. In this paper, the shifted lognormal distribution was employed due to its flexibility. Furthermore, the parameters of the distribution can be explicitly defined in terms of its determined first three central moments, making it convenient for engineering use.

Based on the shifted lognormal distribution, the PDF of the updated conditional reliability index can be formulated as follows

$$f_B\left[\beta^{up}(\boldsymbol{\theta})\right] = \frac{1}{(\beta^{up}(\boldsymbol{\theta}) - \lambda)\sqrt{2\pi}\upsilon} \times \exp\left\{-\frac{1}{2\upsilon^2}\left[\ln\left(\beta^{up}(\boldsymbol{\theta}) - \lambda\right) - \zeta\right]^2\right\} , \tag{15}$$

and its CDF can be derived as:

$$F_B\left[\beta^{up}(\boldsymbol{\theta})\right] = \Phi\left[\frac{\ln\left(\beta^{up}(\boldsymbol{\theta}) - \lambda\right) - \zeta}{\upsilon}\right], \tag{16}$$

where λ, ζ, and υ represent the distribution parameters related to location, scale, and shape respectively. These three distribution parameters for the shifted lognormal distribution can be readily determined by using the first three central moments of the updated conditional reliability index, and the relationship between them can be given as

$$\mu_\beta = \lambda + \exp\left(\zeta + \upsilon^2 / 2\right), \tag{17}$$

$$\sigma_\beta = \sqrt{\left[\exp\left(\upsilon^2\right) - 1\right] \cdot \exp\left(2\lambda + \upsilon^2\right)} , \tag{18}$$

$$\alpha_{3\beta} = \left[\exp\left(\upsilon^2\right) + 2\right]\sqrt{\exp\left(\upsilon^2\right) - 2} . \tag{19}$$

It should be noted that when dealing with positive skewness, Eq. (15) and Eq. (16) can be directly applied. However, in the case of negative skewness, the PDF can be obtained through adopting the absolute value of the skewness and subsequently mirroring the PDF about $\beta^{up}(\boldsymbol{\theta}) = \mu_\beta$, and the CDF can also be determined.

3.3 Comprehensive measurement of the updated conditional failure probability

In this section, a framework is established to comprehensively describe the updated conditional failure probability, which will use the distribution of the updated conditional reliability index to obtain the mean, the quantiles, and even the distribution of the updated conditional failure probability.

3.3.1 Mean of the updated conditional failure probability

Firstly, the distribution of the updated conditional reliability index will be adopted to estimate the updated overall failure probability. Thus, the updated overall failure probability in Eq. (7), can be reformulated as follows:

$$P_F^{up} = E\left[P_f^{up}(\boldsymbol{\theta})\right] = E\left\{\Phi\left[-\beta^{up}(\boldsymbol{\theta})\right]\right\} = \int \Phi\left[-\beta^{up}(\boldsymbol{\theta})\right] \cdot f_B\left[\beta^{up}(\boldsymbol{\theta})\right] d\beta^{up}(\boldsymbol{\theta}) . \tag{20}$$

It can be seen from Eq. (20) that since the distribution of the updated conditional reliability index has been determined, the solution of Eq. (20) is a one-dimensional numerical integration problem and can be readily evaluated through direct integration.

3.3.2 Quantiles of updated conditional failure probability

In many cases, the quantile of the updated conditional failure probability is also an important index for practitioners, not only its mean value. To obtain the quantiles of the updated conditional failure probability, it is essential to first ascertain the corresponding quantiles of the updated conditional reliability index. Following the principle of equal probability, the relationship between the updated CDF of the conditional reliability index and the standard normal random variables can be given as:

$$F_B\left[\beta^{up}(\boldsymbol{\theta})\right] = \Phi(U) . \tag{21}$$

Based on Eq. (16) and Eq. (21), the relationship between the standard normal random variables and the updated conditional reliability index can be deduced as

$$U = \frac{\ln\left(\beta^{up}(\boldsymbol{\theta}) - \lambda\right) - \zeta}{\upsilon} . \tag{22}$$

The quantiles of the updated conditional reliability index for the confidence level α can be represented as follows:

$$F_B\left[\beta^{up}(\alpha)\right] = \Phi(U) = \alpha . \tag{23}$$

According to Eq. (22) and Eq. (23), the quantiles of the updated conditional reliability index for the confidence level α can be derived as follows:

$$\beta^{up}(\alpha) = \exp\left[\Phi^{-1}(\alpha) \cdot \upsilon + \zeta\right] + \lambda . \tag{24}$$

Since the updated conditional failure probability at the α confidence level is a monotonous function of the updated conditional reliability index at the 1- α confidence level, the updated conditional failure probability at the α confidence level can be calculated by:

$$P_f^{up}(\alpha) = \Phi\left[-\beta^{up}(1-\alpha)\right] = \Phi\left\{\exp\left[\Phi^{-1}(1-\alpha) \cdot \upsilon + \zeta\right] + \lambda\right\} . \tag{25}$$

From Eq. (25), it is obvious that this study can provide the updated conditional failure probability at any confidence level α in an explicit form.

3.3.3 PDF of the updated conditional failure probability

To further derive the distribution function of the updated conditional failure probability, it is essential to initially establish the relationship between the standard normal random variables and the updated failure probability. Based on Eq. (6) and Eq. (22), the relationship between the standard normal random variable and the updated conditional failure probability can be derived as follows:

$$U = \frac{\ln\left\{-\Phi^{-1}\left[P_f^{up}(\boldsymbol{\theta})\right]-\lambda\right\}-\zeta}{\upsilon}. \tag{26}$$

Following this, by substituting Eq. (26) into the right-hand side of Eq. (21), the CDF of the updated conditional failure probability can be represented as follows:

$$F_{P_f}\left[P_f^{up}(\boldsymbol{\theta})\right] = \Phi\left(\frac{-\ln\left\{-\Phi^{-1}\left[P_f^{up}(\boldsymbol{\theta})\right]-\lambda\right\}-\zeta}{\upsilon}\right). \tag{27}$$

Finally, by taking the derivative on both sides of Eq. (27), the explicit PDF expression of the updated conditional failure probability can be readily obtained as:

$$f_{P_f}\left[P_f^{up}(\boldsymbol{\theta})\right] = -\frac{\phi\left(\dfrac{\ln\left\{-\Phi^{-1}\left[P_f^{up}(\boldsymbol{\theta})\right]-\lambda\right\}-\zeta}{\upsilon}\right)}{\upsilon\cdot\left\{-\Phi^{-1}\left[P_f^{up}(\boldsymbol{\theta})\right]-\lambda\right\}\phi\left\{\Phi^{-1}\left[P_f^{up}(\boldsymbol{\theta})\right]\right\}}. \tag{28}$$

Similarly, it can be observed from Eq. (28) that the distribution function of the updated conditional failure probability can also be given in the form of an explicit expression.

3.4 Procedure of the proposed method

The computational processes consist of six steps, which are outlined as follows:

Step 1: formulate the calculation equation for the updated conditional reliability index considering the inspection data based on Eqs. (2)- (6).

Step 2: choose the accuracy level q and generate the integration points and weights for uncertain parameters according to SGNI.

Step 3: evaluate the values of the conditional reliability index and likelihood function under each combination of the determined random samples, then the first three raw moments of the updated conditional reliability index can be estimated based on Eq. (10).

Step 4: determine the first three moments of the updated conditional reliability index according to Eqs. (12)-(14).

Step 5: approximate the distribution of the conditional reliability index referring to Eqs. (15)-(16).

Step 6: obtain the quantity of interest of the conditional failure probability, i.e., the mean, the quantiles, and even the distribution of the updated conditional failure probability, with the aid of Eqs. (20)-(28).

4 Numerical example and investigation

In this example, a two degree-of-freedom primary–secondary oscillator subjected to a white noise base acceleration is explored, as depicted in Fig.1. This example has been adapted from the work of Der Kiureghian and De Stefano [24]. The failure of the system is characterized by the surpassing of the peak response of the secondary spring during the duration of the excitation. The expression for the limit state function is as follows:

$$G(\mathbf{X}) = F_s - 3k_s\sqrt{\frac{\pi S_0}{4\zeta_s\omega_s^3}\left[\frac{\zeta_a\zeta_s}{\zeta_p\zeta_s(4\zeta_a^2+\theta^2)+\gamma\zeta_a^2}\frac{\left(\zeta_p\omega_p^3+\zeta_s\omega_s^3\right)\omega_p}{\zeta_a\omega_a^4}\right]}, \tag{29}$$

where F_s represents the force capacity of the secondary spring, S_0 represents the intensity of white noise, k_s and k_p represent the spring stiffnesses, m_p and m_s represent the masses, ζ_p and ζ_s represent the damping rates natural frequencies, $\omega_p = (k_p/m_p)^{1/2}$, $\omega_s = (k_s/m_s)^{1/2}$, $\omega_a = (\omega_p+\omega_s)/2$, $\theta = (\omega_p-\omega_s)/\omega_a$, $\gamma = m_s/m_p$, and $\zeta_a = (\zeta_p+\zeta_s)/2$; and the subscripts p refer to the primary oscillators and subscripts s refer to the secondary oscillators.

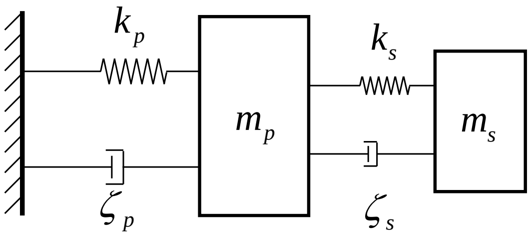

Figure 1: Two degree-of-freedom system

Due to the imperfect state knowledge of the structural parameters m_p, m_s, k_p, and k_s, the mean and standard deviation of these structural parameters are assumed uncertain. Table 1 gives the statistical information of random variables involved in this example as well as the prior information for these uncertain parameters.

Table 1: Statistical information of the random variables

Variables	Distribution	Mean	Standard deviation
m_p	Normal	N (1.0, 0.1)	LN (0.15, 7.5×10^{-3})
m_s	Normal	N (0.05, 0.005)	LN (7.5×10^{-3}, 3.75×10^{-4})
F_s	Lognormal	10	0.1
S_0	Lognormal	100.0	10
k_p	Normal	N (1.0, 0.1)	LN (0.15, 7.5×10^{-3})
k_s	Normal	N (0.01, 0.001)	LN (1.5×10^{-3}, 7.5×10^{-5})
ζ_p	Lognormal	5×10^{-2}	1×10^{-2}
ζ_s	Lognormal	2×10^{-2}	5×10^{-3}

Note: N (μ, σ) represents a normal random variable with a mean of μ and a standard deviation of σ; LN (μ, σ) represents a lognormal random variable with a mean of μ and a standard deviation σ.

In this example, two sets of inspections for structural parameters m_p, m_s, k_p, k_s are collected sequentially, given as $D_{\mathbf{X},1}=\{D_{m_p,1}=1.2, D_{m_s,1}=0.06, D_{k_p,1}=0.9, D_{k_s,1}=0.012\}$ and $D_{\mathbf{X},2}=\{D_{m_p,2}=1.0, D_{m_s,2}=0.04, D_{k_p,2}=1.1, D_{k_s,2}=0.01\}$. Execute the computation procedure of the proposed method as mentioned above, since this example involves eight uncertain parameters, a total of 1105 fixed sample sets were generated through the SGNI (q = 3).

The PDF curves of the posterior reliability index obtained from the proposed method are shown in Fig. 2, compared with the histograms (100,000 samples) generated using MCMC combined with MCS. Likewise, the histograms of the posterior reliability index exhibit a favorable shape (bell-shaped) distribution that is not difficult to fit, and the curves of the posterior reliability index obtained from the proposed method can fit the histograms well.

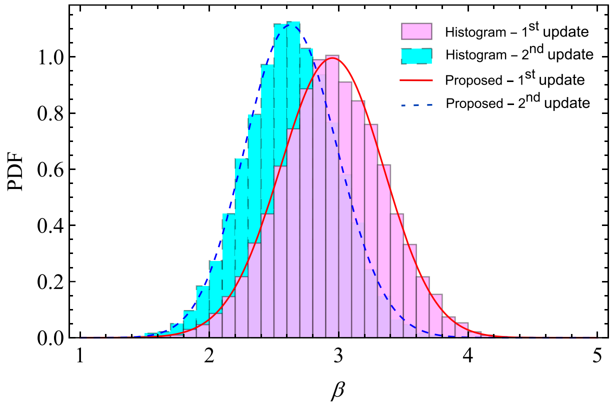

Figure 2: Histogram and PDF of the posterior reliability index

Then, Fig.3 presents the PDF curves of the posterior failure probability derived from the proposed method, compared with the histograms (100,000 samples) of the posterior failure probability determined by MCMC combined with MCS. Fig. 3 clearly indicates that the histograms of the posterior failure probability are heavily skewed to the left, but the curves derived from the proposed method can still agree well with the histograms of the posterior failure probability.

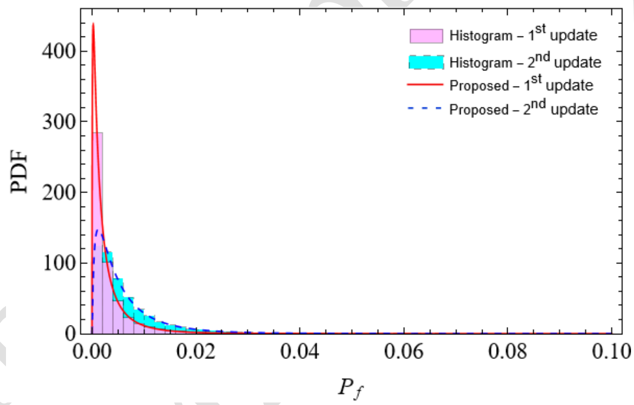

Figure 3: Histogram and PDF of the posterior failure probability

The calculation results and efficiency of the proposed method are presented in Table 2, as well as the results obtained by the combination of MCMC and MCS (100,000 samples). It is indicated in Table 2 that: 1) the results of the mean and quantile of the posterior failure probability are almost the same as those obtained by MCS combined with MCMCS. Consequently, the proposed method exhibits applicability to problems involving multi-dimensional uncertain parameter updates; 2) the proposed method requires only 1105 times of reliability index evaluations within the first update time to obtain the mean and any quantiles of the posterior failure probability. In the subsequent update process, there is no need to perform any additional reliability analysis and acquisition of posterior samples.

Table 2: Results comparison between the proposed method and MCMC+MCS

	Method	NoRE[a]	$\overline{P_f^{up}}$	$P^{up}_{f,0.75}$	$P^{up}_{f,0.9}$
1st update	MCMC+MCS	100,000	3.176×10^{-3}	3.711×10^{-3}	7.612×10^{-3}
	Proposed	1105	3.188×10^{-3}	3.797×10^{-3}	7.547×10^{-3}

			(0.38%)	(2.32%)	(-0.85%)
2^{nd} update	MCMC+MCS	100,000	6.501×10^{-3}	8.260×10^{-3}	1.450×10^{-2}
	Proposed	0	6.472×10^{-3}	8.332×10^{-3}	1.459×10^{-2}
			(-1.14%)	(0.87%)	(0.62%)

Note: [a]Number of reliability index evaluations (NoRE); Values in parentheses are the relative errors of the proposed method compared to MCMC+MCS.

5 Conclusions

This paper introduces an efficient framework to characterize the probabilistic properties of posterior failure probability. The approach leverages sparse grid numerical integration to compute the first three moments of the posterior reliability index, allowing for the reuse of initial analysis results. A shifted lognormal distribution is used to approximate the probability distribution of the posterior reliability index, facilitating the straightforward determination of the mean, quantiles, and probability distribution of the posterior failure probability. A numerical example was examined, leading to the following conclusions:

- The method reuses results from initial evaluations of the reliability index, eliminating the need for additional analyses during multiple updates.
- It separates the impacts of aleatory and epistemic uncertainty on the posterior failure probability and provides analytical solutions for the quantiles and probability distribution using the distribution of the posterior reliability index.
- The results obtained from the proposed method align well with those derived from MCMC combined with MCS, indicating sufficient accuracy.

In this framework, sparse grid numerical integration and the shifted lognormal distribution were recommended as tools to evaluate the first three moments of the posterior reliability index and to approximate its probability distribution. Consequently, the proposed method inherits their limitations. Specifically, the efficiency of the framework may be compromised when dealing with more complex likelihood functions and/or high-dimensional distribution parameters. Additionally, if the posterior reliability index is multimodal, the proposed method is no longer applicable. These limitations can be addressed by incorporating more efficient tools. Future research will focus on handling complex likelihood functions, high-dimensional distribution parameters, and cases where the posterior reliability index is multimodal.

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