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Aleatory and epistemic uncertainty in reliability analysis: An engineering perspective

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ABSTRACT

In engineering applications, aleatory and epistemic uncertainties often coexist and interact. Therefore, accurately modeling these two types of uncertainty is critical for reliability analysis and uncertainty-aware decision making. This is for instance the case when quantifying failure probabilities of engineering structures under consideration of incomplete, insufficient, imperfect, or imprecise data or knowledge. Indeed, in such a case, the failure probability can at best be described using set-theoretical or Bayesian descriptors, rather than as a crisp number to explicitly acknowledge this epistemic uncertainty. However, despite this problem being well-described in theory, we observe that there still exists a gap between the theoretical developments on the one hand, and practical engineering applications of the uncertainty modeling approaches on the other. More precisely, even though the treatment of aleatory and epistemic uncertainty is well understood, they are often still mixed implicitly, or even explicitly in engineering calculations. Therefore, this paper provides a practical engineering guide that should help select the appropriate modeling framework, be it p-boxes, fuzzy probability models, or hierarchical probability approaches, when faced with problems that are affected by both aleatory and epistemic uncertainty. By assessing the type and extent of the information and the purpose of the analysis, this work provides specific recommendations for choosing appropriate modeling methods and presents a comprehensive analysis of failure probability. Additionally, this work highlights the importance of sensitivity analysis in identifying the key parameters that most influence the failure probability. This focus enables engineers to prioritize target data collection, thereby reducing epistemic uncertainty and enhancing the credibility of reliability assessment.

1. Introduction

In assessing the reliability of structures, the identification and characterization of uncertainties play a pivotal role [1]. In practice, uncertainties arise from multiple sources and are typically classified into aleatory and epistemic uncertainty [2,3]. Aleatory uncertainty is a property of the structure under consideration, originating from inherent variability in natural phenomena, such as randomness in material properties, geometric dimensions, environmental conditions, and load characteristics. This type of uncertainty is irreducible; in other words, no matter how much data we manage to collect, it remains unchanged. Epistemic uncertainty, on the other hand, belongs to the observer analyzing the structure. It stems from a lack of knowledge, including factors such as imperfect modeling, simplifications, and limited data, and can be reduced as knowledge improves [4]. In other words, if we would be able to gather more information or data, our state of knowledge could be improved potentially. These two uncertainties originate from different sources, and have fundamentally different meanings. Therefore, it is of key importance to be very clear in one's assumptions when performing reliability analysis, as stressed in the seminal work of Der Kiureghian and Ditlevsen [2]. On top, they often coexist in practical engineering applications: we are looking with imperfect glasses at a random reality. For instance, when considering wind loads on a building, we estimate the long-term statistics of these loads based on historical data. The wind (being aleatory) is modeled based on data; hence our models of the aleatory uncertainty are inherently epistemic—the more data we gather, the better our understanding of the random load becomes. Obviously, in practice, one should consider the relative magnitude of both uncertainties.

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In an ideal scenario where our knowledge is perfect and only aleatory uncertainty remains, probability theory and statistics serve as a well-established and effective tool to model uncertainty. In this case, we can focus on calculating the failure probability, which is a crisp value in principle. To represent aleatory uncertainty in engineering structures, it is usually modeled as a set of continuous random variables $\boldsymbol{X} = (X_1, \ldots, X_n)^T$, where T denotes the transpose. A so-called performance function $g(\boldsymbol{X})$ is then used to describe the state of the structure in terms of $\boldsymbol{X}$. By convention, $g(\boldsymbol{X}) < 0$ denotes failure, $g(\boldsymbol{X}) > 0$ denotes survival, while $g(\boldsymbol{X}) = 0$ denotes the limit state.

The reliability of a structure is usually expressed by its complement, the failure probability P_f, which is defined by the following multi-fold integral:

$$P_f = \int_{g(\boldsymbol{x})\leq 0} f_{\boldsymbol{X}}(\boldsymbol{x})\mathrm{d}\boldsymbol{x}, \tag{1}$$

where $f_{\boldsymbol{X}}(\boldsymbol{x})$ represents the joint probability density function (PDF) of $\boldsymbol{X}$.

Eq. (1) represents the fundamental problem of structural reliability analysis. It is important to point out that in the context of this equation, it is assumed that one has perfect knowledge about the full joint PDF of $\boldsymbol{X}$. However, in practical engineering applications, it is almost impossible for practical and/or economical reasons to have such perfect knowledge. This inherently introduces epistemic uncertainty into the analysis [4]. In general, many sources of epistemic uncertainty exist, including but not limited to epistemic uncertainty on the exact definition of $f_{\boldsymbol{X}}$, but also in the definition of g [2,5]. In this paper, we focus specifically on the case of describing the epistemic uncertainty in $f_{\boldsymbol{X}}$ as it is representative of many engineering cases. To explicitly integrate this uncertainty in our models, a vector of variables $\boldsymbol{\Theta}$ is introduced, transforming the joint PDF of $\boldsymbol{X}$ into a conditional distribution function $f_{\boldsymbol{X}}(\boldsymbol{x}|\boldsymbol{\theta})$. Consequently, the failure probability in Eq. (1) becomes a random variable that is conditional on the distribution parameters $\boldsymbol{\theta}$, referred to as the conditional failure probability, which is expressed as

$$P_f(\boldsymbol{\theta}) = \int_{g(\boldsymbol{x})\leq 0} f_{\boldsymbol{X}}(\boldsymbol{x}|\boldsymbol{\theta})\mathrm{d}\boldsymbol{x}. \tag{2}$$

To address epistemic uncertainty, researchers have explored a wide range of mathematical tools, reflecting the complexity and diverse nature of uncertainty encountered in various situations. Among these, probabilistic approaches provide robust frameworks for uncertainty characterization by leveraging probability theory to model and predict outcomes based on incomplete information [6,7]. These approaches are particularly effective when sufficient empirical data is available to construct reliable probability distribution models, allowing for nuanced insight into structural behaviors. Beyond probabilistic methods, non-probabilistic methods have gained attention for scenarios where empirical data is scarce or qualitative knowledge predominates. Interval models [8–10], for instance, define uncertainty bounds without requiring precise probability distributions, offering a straightforward yet powerful means to handle limited information. Similarly, convex models [11,12] adopt a more generalized approach, accommodating uncertainty by considering sets of possible outcomes confined within a convex domain. Fuzzy sets [10,13,14] add another layer of versatility by allowing for the representation of uncertainty in terms of degrees of membership, capturing the vagueness and ambiguity inherent in many real-world problems. These methods collectively offer engineers and scientists significant flexibility in modeling epistemic uncertainty, enabling the selection of tools tailored to the specific nature and extent of the uncertainty involved. The choice among these approaches typically hinges on the availability of empirical data and the depth of understanding of the underlying structure, as well as the trade-offs between computational complexity and model fidelity. However, despite their strengths, these methods are inherently limited to addressing only the epistemic component of uncertainty. They fail to adequately capture the aleatory uncertainties—those arising from inherent randomness or variability in structures. Consequently, they are often insufficient for comprehensive application in realistic engineering scenarios, where both aleatory and epistemic uncertainties must be accounted for to ensure robust and reliable decision-making. This limitation underscores the need for integrative approaches that can seamlessly model and address the full spectrum of uncertainties present in complex structures.

When modeling aleatory and epistemic uncertainties simultaneously, it is crucial to treat them separately to avoid conflating their distinct sources [5]. This distinction is essential because aleatory uncertainties represent inherent variability, while epistemic uncertainties arise from a lack of knowledge. Blending these fundamentally different interpretations can lead to significant biases in the analysis and misinterpretation of results. Despite its importance, distinguishing between aleatory and epistemic uncertainties during the modeling phase remains challenging [15,16]. To address the difficulty in modeling aleatory and epistemic uncertainties separately, a variety of methods have been developed to address specific scenarios. These include probability-boxes (p-boxes) [17], fuzzy probability [18], hierarchical (Bayesian) probability models [19–21], and others.

The p-box framework serves as an effective and relatively simple tool for addressing scenarios where data is scarce or incomplete, particularly in reliability analysis and risk assessment [17]. By defining upper and lower bounds on cumulative distribution functions (CDFs), the p-boxes capture a range of possible probability distributions rather than committing to a single one. In this framework, the failure probability in Eq. (2) turns out to be an interval. P-boxes are especially advantageous for scenarios involving both aleatory and epistemic uncertainties, particularly when empirical data is limited or the uncertainties are difficult to quantify precisely. For example, in structural engineering, where variability in material properties or loading conditions is poorly understood due to insufficient testing, p-boxes offer a means to account for this uncertainty without making overly optimistic assumptions. However, despite the flexibility that p-boxes offer, they tend to provide only a coarse approximation of epistemic uncertainty [16]. This is because p-boxes impose rigid upper and lower bounds, which may not fully reflect the complex and sometimes ambiguous nature of the underlying uncertainties in real-world engineering problems.

The main issue with traditional interval-probability models such as p-boxes is that they provide no information whatsoever as to what the information looks like within the bounds of the interval [18,22]. However, in some cases, it might be interesting to quantify how sensitive the value of P_f is to the realizations of the epistemic uncertainty within the bounds that are defined on $\boldsymbol{\Theta}$. Fuzzy probabilities provide in this context an excellent extension of the interval-based p-boxes. Following the fuzzy-probability framework, aleatory uncertainty is typically modeled by a probabilistic model, while epistemic uncertainty, arising from the limitations in specifying the probabilistic model, is represented using a fuzzy set. Fuzzy probability theory as such extends the concept of probability bounds, i.e., p-boxes, relaxing the need for precise values or strict boundaries [22–24]. In the context of fuzzy probability theory, the failure probability in Eq. (2) becomes a fuzzy random variable. Fuzzy set theory allows for a smooth transition in the degree of membership of elements, moving from fully belonging to a set to not belonging at all, with the degree of belief or possibility varying gradually. This flexibility makes fuzzy probability particularly powerful in situations involving subjective expert judgments or when data is scarce and ambiguous. More importantly, this flexibility makes fuzzy probabilities mostly useful in the context of assessing the influence of the size of the epistemic uncertainty in the distribution parameters on the epistemic uncertainty in the estimation of P_f. Indeed, since they provide some kind of relationship of values ‘within’ the input intervals with values ‘within’ the P_f interval, a degree of sensitivity can be derived (see Section 3.1). The main drawback of both interval and fuzzy probability models is that neither of them has built-in frameworks to account for additional data to reduce epistemic uncertainty.

In the context of updating prior beliefs on the epistemic uncertainty when more data become available, the Bayesian framework is arguably the most elegant. However, to be able to reframe the updating of our knowledge based on additional data, the epistemic uncertainty has to be recast into a Bayesian framework. This gives rise to so-called hierarchical probabilistic models [25,26]. These models separate aleatory and epistemic uncertainties into two probabilistic layers. Each of these layers has its own interpretation. Typically, the outer layer represents the epistemic uncertainty, and hence is to be interpreted as a subjective (Bayesian) probability. At the same time, the inner layer represents the aleatory uncertainty. In this context, the Bayesian method can be applied to reduce epistemic uncertainty [3,27,28]. As more data becomes available, the subjective influence in model assumptions gradually diminishes. When sufficient data is available for updating and the coefficient of variation (CoV) of the outer (Bayesian) layer is sufficiently small, the parameters of a probabilistic model can be estimated using the expected value of the posterior distribution. This results in a mix of aleatory and epistemic information. Alternatively, the epistemic uncertainty, represented by the posterior distribution, can be made explicit in the results [29], which can facilitate better communication of findings in the context of risk assessment [30].

The modeling and propagation of both aleatory and epistemic uncertainty are well-documented in the literature. Paradoxically, it seems like there still exists a lot of confusion about these concepts, with many authors still mixing up both sources when performing reliability analysis. On top, it seems to be unclear which method is the most appropriate given a certain context. For practical applications, it is challenging to answer this question in general. A realistic mathematical approach can only be formulated by analyzing the nature of the available information in each particular case. To support this analysis and modeling choice, classifications of available information according to defined criteria are usually employed. Moreover, the implications of these uncertainties on the failure probability characteristics have not been thoroughly investigated.

The main objective of this paper is to investigate the gap between advanced theoretical modeling methods for the aleatory and epistemic uncertainties and their practical engineering applications. We aim to provide an engineering perspective on how to rationally apply interval and fuzzy probabilities, along with probability theory, to account for the epistemic uncertainty in reliability analysis problems. Unlike previous studies, our focus is not on introducing new theoretical frameworks but rather on offering insights into how engineers can address aleatory and epistemic uncertainty in calculating failure probabilities, taking into account the type and extent of available information as well as the purpose of the analysis. More specifically, our focus is on three specific models for quantifying the effect of aleatory and epistemic uncertainty on failure probabilities. These models include: p-boxes model, fuzzy probability model, and hierarchical probability model. The remainder of this paper is organized as follows: Section 2 introduces the general framework and formal definition of the specific problem addressed in this study. Section 3 delves into the modeling and propagation of aleatory and epistemic uncertainties, with a focus on the p-boxes model, fuzzy probability model, and hierarchical probabilistic model. Practical considerations and implementation insights are discussed in Section 4. Section 5 presents a classical strip footing problem to demonstrate the practical relevance and application of each method investigated in this study. Finally, the key findings and conclusions of this work are summarized in Section 6. Please note that the mathematical symbols used in this work are defined as the corresponding equations appear in the main text. In addition and for clarity, the notation is also summarized in Appendix.

2. Definition of the general problem

When conducting uncertainty analysis on actual structures, it is essential to first establish a performance function $g(\boldsymbol{X})$ based on the design requirements. This function involves key structural properties such as stiffness and strength, typically related to structural responses like displacement, stress, and strain. For the sake of simplicity, a commonly used performance function that has been investigated in, e.g. [2], is discussed here, given as:

$$g(\boldsymbol{X}) = X_1 - X_2, \tag{3}$$

where X_i $(i = 1, 2)$ are independent random variables, representing the structural resistance and applied load, respectively.

Assuming both X_1 and X_2 are normally distributed, then the failure probability defined in Eq. (1) admits a closed-form expression [31]:

$$P_f = \Phi(-\beta) = \Phi(-\frac{\mu_{X_1} - \mu_{X_2}}{\sqrt{\sigma_{X_1}^2 + \sigma_{X_2}^2}}), \tag{4}$$

where μ_{X_i} and σ_{X_i} are the mean and standard deviation of X_i for $i = 1, 2$, $\Phi(\cdot)$ is the CDF of standard normal variable, and β is the reliability index.

Seting $\mu_{X_1} = 50$, $\mu_{X_2} = 20$, $\sigma_{X_1} = 10$ and $\sigma_{X_2} = 4$, the failure probability incoporating only aleatory uncertainty can be determined as $P_f = \Phi(-\frac{50-20}{\sqrt{10^2+4^2}}) = 2.673 \times 10^{-3}$. Although an analyst might consider this deterministic failure probability acceptable, however, due to imperfect knowledge, epistemic uncertainty arises in estimating the exact values of distribution parameters for X_1 and X_2. In particular, both the means and standard deviations of X_1 and X_2, denoted by parameter vector $\boldsymbol{\Theta} = [\mu_{X_1}, \mu_{X_2}, \sigma_{X_1}, \sigma_{X_2}]$, are subject to uncertainty. As a result, the failure probability, which was previously treated as deterministic, now becomes a variable that depends on the uncertain distribution parameters $\boldsymbol{\Theta}$. This shift introduces epistemic uncertainty into the assessment of failure probability. The conditional failure probability, given a specific realization of $\boldsymbol{\Theta} = \boldsymbol{\theta}$, retains the same functional form as in the deterministic case:

$$P_f(\boldsymbol{\theta}) = \Phi(-\beta(\boldsymbol{\theta})) = \Phi(-\frac{\mu_{X_1} - \mu_{X_2}}{\sqrt{\sigma_{X_1}^2 + \sigma_{X_2}^2}}), \tag{5}$$

where $\beta(\boldsymbol{\theta})$ is the corresponding conditional reliability index.

Many different models have been available for representing both aleatory and epistemic uncertainties in different situations based on the available information. The choice of appropriate models to model and propagate these uncertainties is left open and will be discussed in the rest of the paper.

3. Modeling and propagation of aleatory and epistemic uncertainties

3.1. Probability box (p-box) model

In some practical engineering applications, due to lack of knowledge, the probability distribution of random variables is unknown but can be constrained. In this case, both distribution-free and parametric p-boxes can be constructed based on available information [17]. Under the p-box framework, the true but unknown CDF $F_{X_i}(x_i)$ of the random variable X_i (for $i = 1, 2$) in Eq. (3) is assumed to lie within the interval $[\underline{F}_{X_i}(x_i) \leq F_{X_i}(x_i) \leq \overline{F}_{X_i}(x_i) | x_i \in D_{X_i}]$ [32,33], where $\underline{F}_{X_i}(x_i)$ and $\overline{F}_{X_i}(x_i)$ denote the lower and upper bounds CDF of X_i, respectively. The p-box concepts define lower probability operator $\underline{\mathbb{P}}$ and upper probability operator $\overline{\mathbb{P}}$ for events $\{X_i \leq x_i\} = (-\infty, x_i] \cap D_{X_i}$, such that $\underline{\mathbb{P}}(X_i \leq x_i) = \underline{F}_{X_i}(x_i)$ and $\overline{\mathbb{P}}(X_i \leq x_i) = \overline{F}_{X_i}(x_i)$. This formulation generates a credal set of probability measures. The p-box framework effectively handles the epistemic uncertainty, such as that caused by incomplete information on $F_{X_i}(x_i)$, by assigning an interval $[\underline{F}_{X_i}(x_i), \overline{F}_{X_i}(x_i)]$ for each value of X_i [33]. If high-quality information is available across the full range of possible values for X_i this interval becomes narrow, and the p-box approximates a crisp, deterministic CDF. However, when less information is available, the bounds widen to reflect weaker confidence in the resulting estimates.

For situations where there is no prior knowledge of a specific distribution family, the distribution-free p-boxes are recommended [34]. This is the most general type of p-boxes, which allows for the most flexibility in the modeling of parameters subject to aleatory and epistemic uncertainties since any non-decreasing and right-continuous function that ranges from 0 to 1 within these bounds is admissible. Since distribution-free p-boxes are so general in their definition, also CDFs that are questionable from a physical perspective are explicitly included in the definition. That means the free p-boxes are defined only by the upper and lower bounds of the CDF, and the true CDF can have any shape as long as it satisfies the characteristics of the general CDF and lies within the boundaries of the free p-boxes. In contrast, if the distribution family is known but the distribution parameters $\boldsymbol{\Theta}$ of the CDF $F_{\boldsymbol{X}}(\boldsymbol{x}|\theta)$ are uncertain, a parametric p-box is preferable. This method confines the possible CDFs by restricting the parameters within known intervals. Parametric p-boxes clearly distinguish between aleatory uncertainty, captured by the known distribution family, and the epistemic uncertainty, represented by parameter intervals [35]. Based on the definition, the upper and lower bounds of the parametric p-box for components X_i (for $i = 1, 2$) in Eq. (3) can be calculated by finding the minimum and maximum values of its CDF within the interval of parameter uncertainty. These bounds are computed by:

$$\underline{F}_{X_i}(x_i) = \min_{\theta_i \in D_{\theta_i}} F_{X_i}(x_i \mid \boldsymbol{\theta}_i), \tag{6}$$

$$\overline{F}_{X_i}(x_i) = \max_{\theta_i \in D_{\theta_i}} F_{X_i}(x_i \mid \boldsymbol{\theta}_i), \tag{7}$$

where $F_{X_i}(x_i|\boldsymbol{\theta}_i)$ denotes the conditional CDF of X_i, and D_{θ_i} denotes a hyper-rectangular domain of θ_i. The derivative of this CDF with respect to X_i is $f_{X_i}(x_i|\boldsymbol{\theta}_i)$ in case it can be differentiated. Note that the distribution parameters $\boldsymbol{\theta}_i = [\mu_{X_i}, \sigma_{X_i}]$ are uncertain and can be characterized by the interval model $\boldsymbol{\theta}_i \in [\underline{\boldsymbol{\theta}_i}, \overline{\boldsymbol{\theta}_i}]$, where $\underline{\boldsymbol{\theta}_i} = [\underline{\mu}_{X_i}, \underline{\sigma}_{X_i}]$ and $\overline{\boldsymbol{\theta}_i} = [\overline{\mu}_{X_i}, \overline{\sigma}_{X_i}]$ are the lower bound and upper bound, respectively.

Fig. 1 illustrates both distribution-free p-box and parametric (Gaussian) p-box used to model the CDFs of X_1 and X_2. Specifically, the parametric p-box is derived from Gaussian distributions: $\Phi(x_1; \mu_{X_1}, \sigma^2_{X_1})$, where $\mu_{X_1}=$ [45,55] and $\sigma_{X_1}=$ [9,11], and $\Phi(x_2; \mu_{X_2}, \sigma^2_{X_2})$, where $\mu_{X_2}=$ [18,22] and $\sigma_{X_2}=$ [3,5]. Note that while the Gaussian p-box relies on these parameter bounds, the distribution-free p-box is constructed solely based on bounding constraints on CDFs without assuming a specific parametric form. It can be seen from Fig. 1 that the distribution-free p-box considers only the upper and lower CDFs, and any CDF that complies with these bounds is admissible. The parametric p-box on the other hand, imposes additional constraints on admissible distribution functions, for instance, by defining a family of distribution functions.

When considering a reliability problem subject to p-boxes as input, the failure probability is not a single value due to the range of possible CDFs. Instead, it becomes a set-valued quantity, where both the lower bound $\underline{P}_f$ and the upper bound $\overline{P}_f$ of the failure probability are of great concern in most engineering applications [17]. For the case of free p-boxes, the random vector $\boldsymbol{C}$ with length K that is uniformly distributed in the unit-hypercube domain $D_C = [0, 1]^K$ is introduced. The lower- and upper bounds of the failure probability as shown in Eq. (2) can be defined as follows [36,37]:

$$\underline{P}_f = \mathbb{P}\left(\overline{g}(\boldsymbol{C}) \le 0\right) = \mathbb{P}\left(\max_{X_i \in [\underline{X}_i, \overline{X}_i]} g(\boldsymbol{X}) \le 0\right), \tag{8}$$

$$\overline{P}_f = \mathbb{P}\left(\underline{g}(\boldsymbol{C}) \le 0\right) = \mathbb{P}\left(\min_{X_i \in [\underline{X}_i, \overline{X}_i]} g(\boldsymbol{X}) \le 0\right). \tag{9}$$

where $\mathbb{P}(\cdot)$ denotes the probability operator. For each component X_i ($i = 1, 2$), the lower and upper bounds are defined as $\underline{X}_i = \overline{F}^{-1}_{X_i}(C_i)$ and $\overline{X}_i = \underline{F}^{-1}_{X_i}(C_i)$, respectively, where C_i represents the value of the CDF associated with the independent random variable X_i and is the ith component of the vector $\boldsymbol{C}$. Here, $\overline{F}^{-1}_{X_i}$ and $\underline{F}^{-1}_{X_i}$ are the inverse of the lower and upper bounds of the CDF, which map the given CDF value C_i to the corresponding bounds of X_i.

The lower and upper bounds of the failure probability, calculated using Eqs. (8)–(9) with the free p-boxes specified in Fig. 1, are obtained as $\hat{\underline{P}}_f = 4.340 \times 10^{-5}$ and $\hat{\overline{P}}_f = 2.974 \times 10^{-2}$. The main contributors to the total computational costs involved in Eqs. (8)–(9) are: (i) evaluating the limit-state function, (ii) performing the 2-dimensional optimization in the model conversion, and (iii) estimating the failure probability boundaries through 10^6 Monte Carlo Sampling (MCS). Since the limit-state function in Eq. (3) is relatively inexpensive to evaluate, the current approach remains efficient. However, for the general scenarios, advanced techniques such as those proposed in [37–39] can be employed to enhance efficiency.

For parametric p-box cases, the distribution parameters are interval-valued. If the conditional failure probability is expressed in the form of Eq. (5), the lower and upper bounds of the failure probability can be defined as:

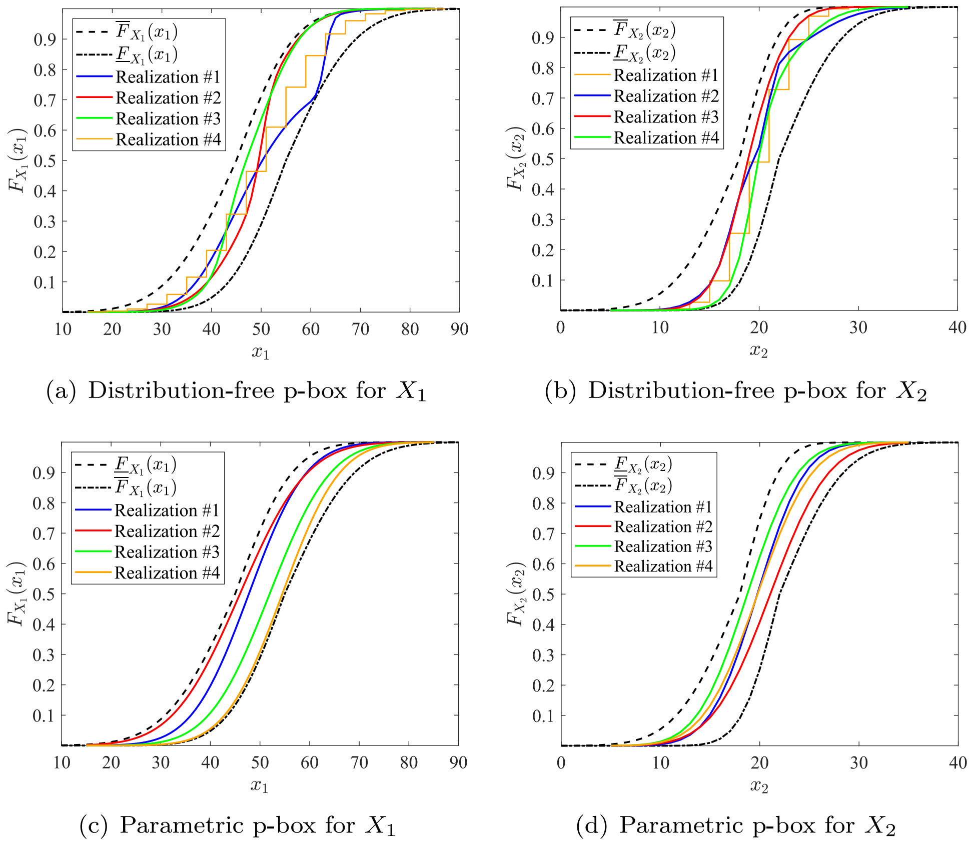
$$\underline{P}_f = \min_{\theta}(P_f(\boldsymbol{\theta})) = \min_{\theta}\left(\Phi\left(-\frac{\mu_{X_1} - \mu_{X_2}}{\sqrt{\sigma^2_{X_1} + \sigma^2_{X_2}}}\right)\right), \tag{10}$$

$$\overline{P}_f = \max_{\theta}(P_f(\boldsymbol{\theta})) = \max_{\theta}\left(\Phi\left(-\frac{\mu_{X_1} - \mu_{X_2}}{\sqrt{\sigma^2_{X_1} + \sigma^2_{X_2}}}\right)\right). \tag{11}$$

The lower and upper bounds of the failure probability, calculated using Eqs. (10)–(11) with the parametric p-boxes specified in Fig. 1, are determined as $\hat{\underline{P}}_f = 5.352 \times 10^{-5}$ and $\hat{\overline{P}}_f = 2.758 \times 10^{-2}$, respectively. The estimation of boundary values of the failure probability, as shown in Eqs. (10)–(11), involves two key steps: (i) calculating the conditional failure probabilities $P_f(\boldsymbol{\theta})$, and (ii) performing optimization on 10^6 MCS samples of $P_f(\boldsymbol{\theta})$. Since the conditional failure probability $P_f(\boldsymbol{\theta})$ given in Eq. (5) is explicitly expressed, the computational effort required for Eqs. (10)–(11) is relatively manageable. For more general cases, advanced techniques, such as those proposed in [37,40,41], can be employed to reduce computational costs.

The width of the failure probability interval reflects the level of epistemic uncertainty: wider intervals indicate greater epistemic uncertainty on the failure probability estimate. When the analyst has access to high-quality information across the full range of X_i to establish a description of the uncertainty through p-boxes, the interval $[\underline{P}_f, \overline{P}_f]$ will narrow, and thus the failure probability closely approximates to a crisp value. In contrast, limited or poor information might lead to a wider interval, indicating lower confidence in the failure probability estimate. In the previous sentence, the word 'might' is of importance: it might be that only limited information on some input quantities is available, whereas this does not affect the failure probability interval too much. As such, the interval that is calculated on the failure probability can act as a measure of the trustworthiness of the failure probability estimation given the epistemic uncertainty.

In most engineering applications, the upper bound of the failure probability is of primary concern, as it represents the worst-case scenario given the epistemic uncertainty. In case we are only interested in this upper bound, such as in estimating the failure probability of existing structures, wide failure probability intervals might still be acceptable. However, for structural design, relying solely on the upper bound of the failure probability can be overly conservative. While this approach may ensure safety, it does not support optimal decision-making. To achieve a balance between safety and cost-effectiveness in structural design, it is essential to consider the entire interval of the failure probability. In this section, the bounds for the failure probability both from free p-boxes and parametric p-boxes vary within 10^{-5} and 10^{-2}, indicating that the epistemic uncertainty interval is too wide for the analysis to be meaningful. Therefore, it becomes evident that more analysis of the epistemic uncertainty is needed. In such cases, it becomes crucial to determine whether additional data collection is



(a) Distribution-free p-box for X_1 (b) Distribution-free p-box for X_2

(c) Parametric p-box for X_1 (d) Parametric p-box for X_2

Fig. 1. Illustration of distribution-free and parametric p-boxes.

needed and, if so, to identify the specific types of data that would be most beneficial. Fuzzy probability analysis can assist in identifying whether the epistemic uncertainty can be reduced, and this approach will be discussed in the next subsection.

Furthermore, identifying which uncertain parameters have the greatest influence on the probability of failure P_f, is crucial to guide practitioners in collecting data. Sensitivity analysis has emerged as a powerful tool [42–44] to select important uncertain parameters that influence structural performance. However, analyzing the sensitivity of the failure probability valued in the interval remains a challenge. Traditional approaches, such as computing the gradient of failure probability at interval boundaries, offer valuable local insights but fall short of offering a comprehensive understanding of the problem. To address these limitations, fuzzy probabilistic methods have been introduced, providing a broader and more holistic perspective on the uncertainties influencing the structure. These methods not only enhance understanding of the overall trends in failure probability across the uncertainty landscape but also identify key areas where additional data collection can effectively reduce uncertainty. Simultaneously, Bayesian methods serve as a complementary tool to refine probability estimates and enhance the accuracy of predictions. Together, these methodologies offer a robust framework for understanding epistemic uncertainty, enabling well-informed decisions in both data collection and structural design.

3.2. Fuzzy probability model

When precise identification of bounds in interval models needs to be relaxed, fuzzy set theory [13] provides an effective framework by introducing the gradual assignment of membership values, accommodating broader uncertainties. In this context, the uncertain distribution parameters $\boldsymbol{\Theta}$ in Eq. (5) are represented as fuzzy sets. Specifically, the fuzzy set $\widetilde{\theta}_{i,l}$ associated with the lth distribution parameter of the ith random variable is defined as follows:

$$\widetilde{\theta}_{i,l} = \{(\theta_{i,l}, \mu_{\widetilde{\theta}_{i,l}}(\theta_{i,l})) : (\theta_{i,l} \in \Theta_{i,l}) \wedge (\mu_{\widetilde{\theta}_{i,l}}(\theta_{i,l}) \in (0,1])\}, \quad i = 1,2; \quad l = 1,2. \tag{12}$$

where $\mu_{\widetilde{\theta}_{i,l}}(\theta_{i,l})$ represents the membership function that assigns a degree of membership to each value $\theta_{i,l}$. The value of $\mu_{\widetilde{\theta}_{i,l}}(\theta_{i,l})$ reflects how strongly $\theta_{i,l}$ belongs to the fuzzy set. Specifically, a membership value close to 1 indicates a high degree of membership, while values approaching 0 indicate lower degrees of membership.

These membership values can represent different semantics depending on the application, such as similarity, preference, or uncertainty [45,46]. The semantics of similarity quantify the proximity of elements within a fuzzy set and are typically used in fuzzy control. Similarity can also be used for interpretation in relation to sensitivity analysis. The semantics of preference represent the intensity of favoring certain elements and are useful for design purposes. While the interpretation of membership as belief refers to the semantics of uncertainty for the interpretation of a fuzzy set. Using this semantics matches the context of reliability analysis, which is the focus of this study. To accommodate these diverse semantics, the aggregation rules for fuzzy sets can be chosen differently. Specifically, t-norms and t-co-norms can be employed. The selection of a particular t-norm and its dual t-conorm is justified by the context in which the fuzzy sets are applied [46,47]. In this study, we focus on processing fuzzy information through structural computations rather than fuzzy logic itself. The interpretation of membership values as uncertainty (epistemic possibility) leads to the natural choice of the min-operator as a specific case of a t-norm and the max-operator as a corresponding t-co-norm. This results in the min–max operator and the extension principle [47], which are standard in fuzzy information processing. In engineering computations, the original

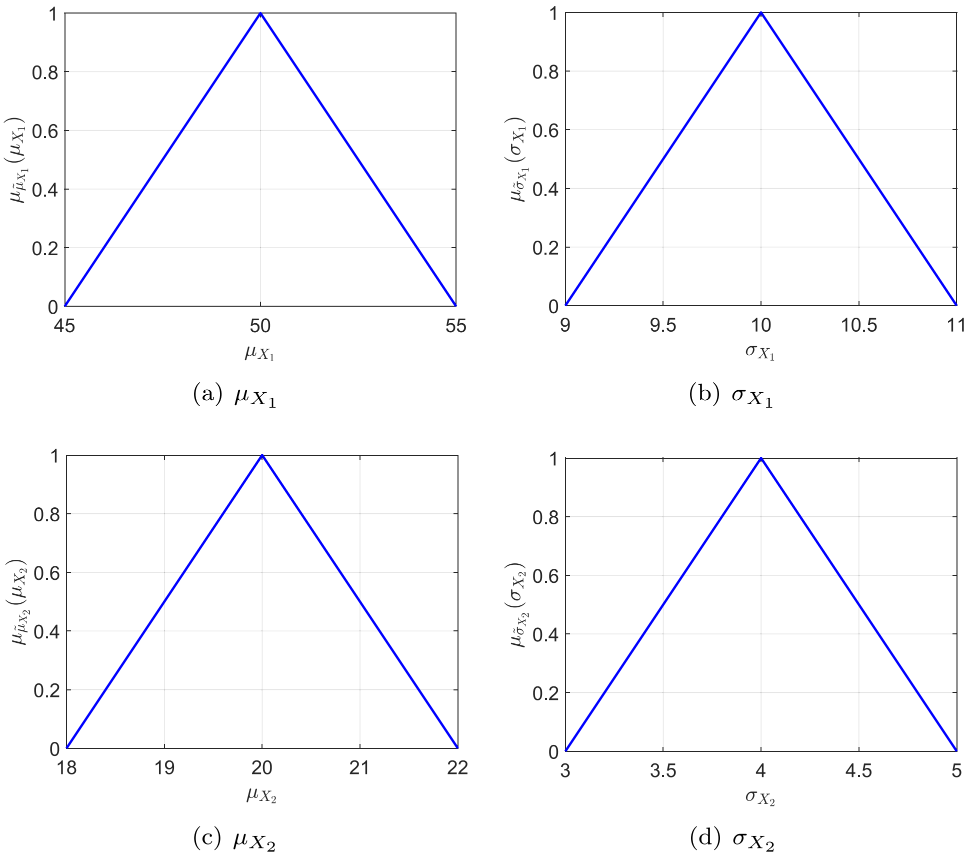


Fig. 2. Membership functions of distribution parameters.

form of the extension principle is unsuitable for numerical algorithms because it requires handling continuous operations, which are computationally intensive and challenging to implement. To address this issue, an alternative method based on α-discretization was proposed. This method approximates the results of the extension principle by decomposing fuzzy sets into several discrete α-levels, while retaining consistency in the use of the min–max operator to ensure computational accuracy and practicality [24].

Various types of membership functions exist, including trapezoidal, Gaussian, and triangular functions [48]. Among them, the triangular membership function is commonly used in practical engineering due to its simplicity and ease of implementation. In this section, triangular membership functions are used to model the distribution parameters. The membership is characterized by its lower bound $\underline{\theta}_{i,l}$ and upper bound $\bar{\theta}_{i,l}$ for which $\mu_{\widetilde{\theta}_{i,l}}(\underline{\theta}_{i,l}) = \mu_{\widetilde{\theta}_{i,l}}(\overline{\theta}_{i,l}) = 0$; and $\theta_{i,l,\mu}$ for which $\mu_{\widetilde{\theta}_{i,l}}(\theta_{i,l,\mu}) = 1$. This data is compactly represented as $\langle \underline{\theta}_{i,l}, \theta_{i,l,\mu}, \overline{\theta}_{i,l} \rangle$. For a given membership level $\alpha \in (0,1]$ of a convex membership function [18], an interval $\theta^I_{i,l,\alpha}$ can be extracted, with bounds $\underline{\theta}_{i,l,\alpha}$ and $\overline{\theta}_{i,l,\alpha}$. This interpretation aligns with the concept that fuzzy sets with convex membership functions can be seen as a collection of intervals indexed by the membership level α [10]. Fig. 2 illustrates the membership functions of distribution parameters $(\mu_{X_1}, \sigma_{X_1}, \mu_{X_2}, \sigma_{X_2})$ using triangular membership functions.

Then the aleatory uncertainty in the random variable X_i in Eq. (3) can be described by its probability distributions, while the epistemic uncertainty in the distribution parameters $\boldsymbol{\Theta}_i$ is modeled as fuzzy sets. This approach combines fuzzy set theory and probability theory [18, 23], resulting in what is known as fuzzy probabilities [23]. Thus, the random variables X_i in Eq. (3) become fuzzy random variables $\tilde{X}_i$, defined by their fuzzy CDF $\tilde{F}_{X_i}(x_i|\theta_i)$ given as:

$$\tilde{F}_{X_i}(x_i|\theta_i) = \{(F_{X_i}(x_i|\theta_i), \mu(F_{X_i}(x_i|\theta_i))) : F_{X_i}(x_i|\theta_i) \in \mathbb{F}, \mu(F_{X_i}(x_i|\theta_i)) \in (0,1]\}, \tag{13}$$

where $\mathbb{F}$ denotes the set of admissible CDFs.

For a specific α-level ($\alpha \in (0,1]$), the fuzzy CDF $\tilde{F}^{\alpha}_{X_i}(x_i|\theta_i)$ can be expressed as:

$$\tilde{F}^{\alpha}_{X_i}(x_i|\theta_i) = \{F_{X_i}(x_i|\theta_i) \in \mathbb{F} : \mu(F_{X_i}(x_i|\theta_i)) \geq \alpha\}. \tag{14}$$

The lower and upper bounds of this CDF for each α-level can be defined as

$$\underline{F}^{\alpha}_{X_i}(x_i) = \min_{\theta_i \in D_{\theta_i}} \left\{ \tilde{F}^{\alpha}_{X_i}(x_i|\theta_i) \right\}, \tag{15}$$

$$\overline{F}^{\alpha}_{X_i}(x_i) = \max_{\theta_i \in D_{\theta_i}} \left\{ \tilde{F}^{\alpha}_{X_i}(x_i|\theta_i) \right\}, \tag{16}$$

where D_{θ_i} denotes the domain of distribution parameters θ_i. Fig. 3 shows the fuzzy probability model of X_i for illustration of the concept.

Because of the fuzziness in the probabilistic model, the failure probability also becomes a fuzzy variable [49,50]. The α-level optimization method [14,49] provides a potential approach for determining the membership function of failure probability P_f. This method involves performing optimization at a series of discrete membership levels α. For a given α-level, the uncertainty in the distribution parameters is treated as an interval θ^I_{α}, which allows us to calculate the interval bounds for the failure probability. These bounds, $\underline{P}^{\alpha}_f$ and $\overline{P}^{\alpha}_f$, are given by:

$$\underline{P}^{\alpha}_f = \min_{\theta \in \theta^I_{\alpha}} \left(P_f(\theta) \right), \tag{17}$$

$$\overline{P}^{\alpha}_f = \max_{\theta \in \theta^I_{\alpha}} \left(P_f(\theta) \right), \tag{18}$$

where θ^I_{α} denotes the hyper-rectangle defined by the individual intervals.

The pair of Eqs. (17) and (18) defines an interval analysis at a specific α-level, where the bounds are obtained through an optimization

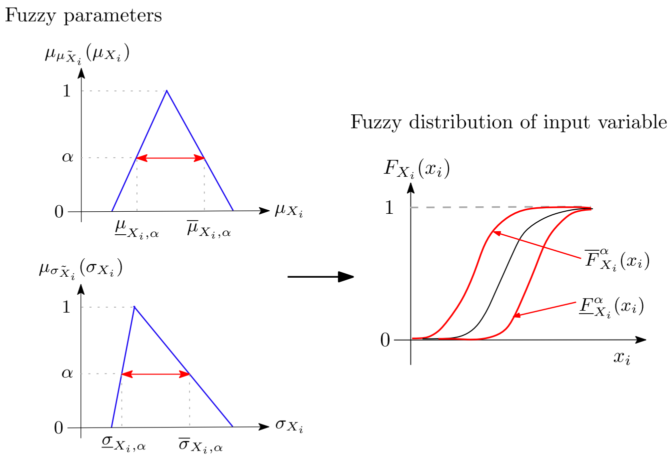


Fig. 3. Illustration of a fuzzy probability model.

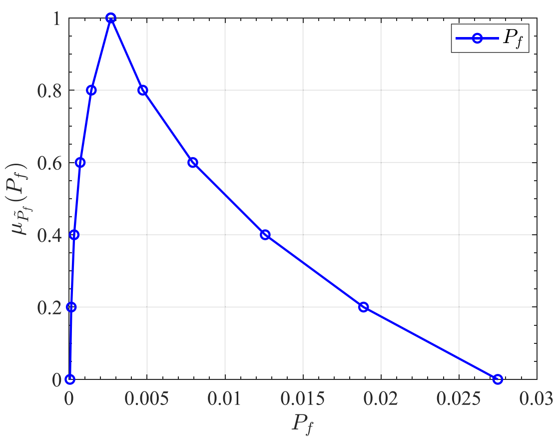


Fig. 4. Membership function of failure probability.

process. By repeating this analysis for different membership levels α, a collection of failure probability intervals $P^I_{f,\alpha}$ is generated, which provides a discrete approximation of the membership function corresponding to the failure probability [49]. The membership function of failure probability is shown in Fig. 4. It is computed using the same parameter intervals for the distribution as those specified in Fig. 1, while incorporating the membership values presented in Fig. 2.

Several important observations can be drawn from Fig. 4. First, the value of the failure probability P_f, given as 2.673×10^{-3} for a membership value equal to 1 is the same as that reported in Section 2. This is consistent with the definition of the membership functions assumed for the distribution parameters in Fig. 2: for a membership level equal to 1, there is no epistemic uncertainty about the distribution parameters and hence, the failure probability becomes a crisp number. Second, the support of the failure probability, given as $[\underline{\hat{P}}_f = 5.352 \times 10^{-5}, \overline{\hat{P}}_f = 2.749 \times 10^{-2}]$, for values of membership approaching 0 is nearly the same as that of Section 3.1 associated with the parametric p-box model. This is again a consequence of the definition of the membership functions in Fig. 2 as well as the intervals chosen in Section 3.1. Third, by observing the shape of the membership function associated with the failure probability in Fig. 4, it is concluded that it is highly sensitive to the level of epistemic uncertainty.

In this process, each failure probability value in Eq. (5) under a different combination of distribution parameters can be directly determined, with the imprecision in the fuzzy variable $\tilde{X}_i$ mapped onto the result. Specifically, the analysis employs α-discretization, where each α-level set $\theta^I_{i,l,\alpha}$ of the fuzzy variable $\tilde{\theta}_{i,l}$ corresponds to an interval parameter of a probability distribution, thereby can be termed as the p-boxes. For a given α-level, the corresponding analysis produces an interval for the failure probability associated with that level. By repeating this p-box analysis across multiple α-levels, a nested set of intervals for P_f is generated, resulting in the fuzzy failure probability. Notably, this approach explicitly determines the crisp set of failure probabilities for varying membership levels, offering significant practical value. It provides critical insights into the sensitivity of the failure probability to the imprecision in distribution parameters. Additionally, since the failure probability is treated as a fuzzy variable, it inherently incorporates epistemic uncertainty in the distribution parameters. This suggests an external approach to the problem, considering the uncertainty in the available information about the structure.

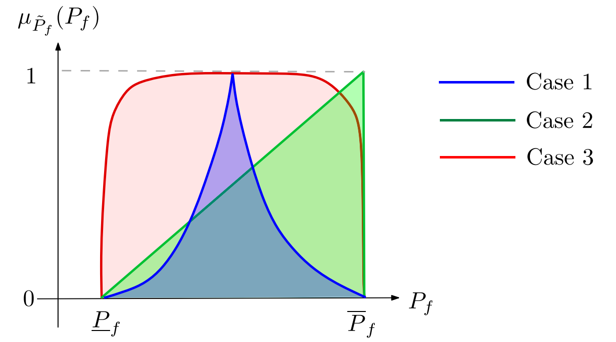


Fig. 5. Illustration of different membership functions associated with failure probability. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

In addition, Fig. 5 shows different membership functions associated with the failure probability considering the fuzzy distribution parameters. These cases are discussed in the following.

- The blue line (case 1) illustrates that as the membership levels α increase, the epistemic uncertainty associated with the failure probability decreases. This indicates that the membership functions associated with the fuzzy distribution parameters significantly influence the final failure probability. Such a result suggests that the effect of epistemic uncertainty could be drastically reduced by collecting additional data regarding the distribution parameters. In other words, such additional data would allow to reduce epistemic uncertainty on the distribution parameters and ultimately, reduce the epistemic uncertainty associated with the failure probability. Therefore, and in addition to collecting more data, it would be advisable to switch to a hierarchical probability model, as such model allows applying Bayes' theorem for reducing epistemic uncertainty objectively based on prior beliefs regarding distribution parameters and additional data collected. Moreover, it is observed that the membership function associated with the failure probability shown in Fig. 4 falls precisely into the case described above. Therefore, for the numerical example considered in this paper, it makes perfect sense to switch to a hierarchical probability model. Such an option is explored further in Section 3.3 of this paper.
- The green line (case 2) shows that as the membership levels α increase, the upper bound of failure probability remains unchanged, suggesting the upper bound is not affected by a reduction of the epistemic uncertainty. Therefore, for this case, if the practitioner is only concerned with the upper bound of the failure probability, collecting more information to reduce the epistemic uncertainty may not be necessary.

- The red line (case 3) demonstrates that as the membership level α increases, both the lower and upper bounds of the failure probability show minimal changes, implying that the membership function of the fuzzy distribution parameters has little effect on the failure probability. Such behavior could reveal different issues. For example, there may be a strong non-linear interaction between the physics of the problem (which encapsulated into the performance function $g(\boldsymbol{X})$) and the uncertainty model (contained in $f_{\boldsymbol{X}}(\boldsymbol{x}|\boldsymbol{\theta})$ and $\mu_{\tilde{\theta}_{i,l}}$). Such an interaction may be found in, e.g., dynamical problems, see e.g., [51]. Another example could be that there are sources of uncertainty which have not been properly accounted for and that prevent reducing the effect of epistemic uncertainty even under different membership levels of the distribution parameters. Such an issue could reveal that the uncertainty model is incomplete; see e.g., [52].

The preceding paragraph highlights that fuzzy probability is most useful for identifying an overall trend regarding the interaction between epistemic and aleatory uncertainties. Indeed, with a fuzzy representation, it becomes possible to assess how reducing the epistemic uncertainty associated with the distribution parameters affects the values of the failure probability. Specifically, if $\left[\underline{F}^{\alpha_i}_{X_i}(x_i), \overline{F}^{\alpha_i}_{X_i}(x_i)\right] \subseteq \left[\underline{F}^{\alpha_j}_{X_i}(x_i), \overline{F}^{\alpha_j}_{X_i}(x_i)\right]$ for $x_i \in D_{X_i}$, then it follows that $\left[\underline{P}^{\alpha_i}_{f}, \overline{P}^{\alpha_i}_{f}\right] \subseteq \left[\underline{P}^{\alpha_j}_{f}, \overline{P}^{\alpha_j}_{f}\right]$ for $0 \le \alpha_j \le \alpha_i \le 1$.

The characteristics of this fuzzy probabilistic analysis can help identify the sensitivity of the failure probability to the imprecision in the probabilistic model specifications. When the interval size of the failure probability increases significantly with a moderate increase in the size of the input parameter intervals, it indicates that the failure probability is highly sensitive to these uncertainties. In this case, the membership function of the failure probability will exhibit extensibility or long, flat tails.

From an engineering perspective, it is important to pay special attention to the input model options that lead to large intervals of the failure probability [53,54]. However, fuzzy probability by itself does not pinpoint which parameters of the vector $\boldsymbol{\Theta}$ are the most influential ones. It just provides a notion on the overall sensitivity. Moreover, robust analysis under fuzzy uncertainty is critical in structural design, as it addresses the challenge to ensure optimal performance under environmental fluctuations and parameter variability [55,56]. To apportion the effect of the epistemic uncertainty associated with the distribution parameters onto the failure probability, it is necessary to apply special sensitivity indexes that cope with fuzzy uncertainty. These indices can help enhance the understanding of which parameters drive failure probability and contribute to the design of robust systems that can withstand variability. Although several methods exist for sensitivity analysis under fuzzy uncertainty (see, e.g. [57,58]), specific tools to quantify the sensitivity of failure probability with respect to epistemic uncertainty are (to authors' best knowledge) still to be developed. Fuzzy numbers, a specialized form of random sets, offer a promising avenue for creating such tools. By capturing the inherent imprecision in model parameters, fuzzy numbers facilitate a more detailed understanding of how variations in epistemic uncertainty influence failure probability. This approach provides a finer analysis of uncertainty's impact, ultimately supporting more informed decision-making in risk assessment and structural design.

3.3. Hierarchical probability model

If expert knowledge, prior assumptions, and relevant data are sufficient to model epistemic uncertainty using probability models, then both aleatory and epistemic uncertainty can be addressed within a framework of probability theory, known as the hierarchical probability model [21,25,59]. It is worth emphasizing the critical distinction between the Bayesian (subjective) representation of epistemic uncertainty — reflecting beliefs about unknown parameters — and the frequentist (objective) representation of aleatory uncertainty, which captures inherent randomness in a structure. In the hierarchical probability model, the distribution parameters $(\mu_{X_1}, \mu_{X_2}, \sigma_{X_1}, \sigma_{X_2})$ in Eq. (5) are considered as random variables within the outer loop, while aleatory uncertainty is characterized by the random variable X_i $(i = 1, 2)$ with their PDF $f_{\boldsymbol{X}}(\boldsymbol{x}|\boldsymbol{\theta})$ in the inner loop. Consequently, the failure probability shown in Eq. (5) also becomes a random variable. The failure probability in Eq. (5) under each set of distribution parameters can be directly calculated as it is given in an explicit expression. For more general cases, advanced techniques such as the first- and second-order reliability methods [60,61], moment methods [31], and simulation methods [62] need to be employed to evaluate the failure probability values under each set of distribution parameters. Fig. 6 presents the histogram of the failure probability, as defined by Eq. (5), calculated using MCS with 1,000,000 samples. The probabilistic characteristics for prior distribution of the distribution parameters used in this analysis are outlined in Table 1.

Table 1
Probabilistic information for prior distribution of the distribution parameters.

Parameter	Distribution	Mean value	CoV
μ_{X_1}	Lognormal	50	0.2
μ_{X_2}	Lognormal	20	0.2
σ_{X_1}	Lognormal	10	0.2
σ_{X_2}	Lognormal	4	0.2

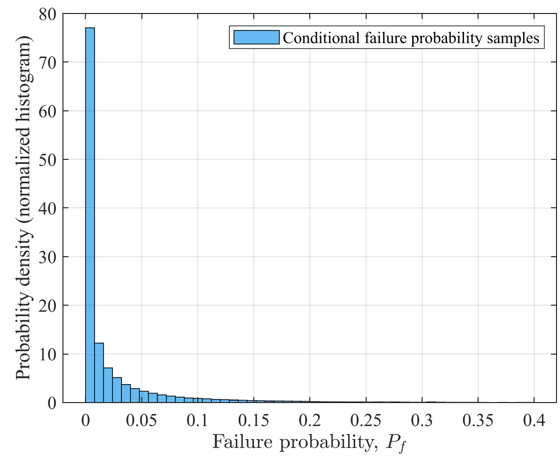


Fig. 6. Histogram of the failure probability.

As illustrated in Fig. 6, the failure probability is consistently constrained within the range of 0 to 1 and exhibits an L-shaped distribution. From the histogram of failure probability, key quantities can be determined based on the practitioner's needs, including the mean, CoV, quantiles, confidence intervals, and probability distributions [63].

The mean value of the failure probability (given in Eq. (5)), referred to as the expected failure probability P_F, can be computed using MCS as shown in the following equation:

$$P_F = \int_{\theta} P_f(\theta) f_{\Theta}(\theta) d\theta \approx \frac{1}{M} \sum_{j=1}^{M} P_f(\theta_j), \tag{19}$$

where M represents the number of samples.

Using the 1,000,000 samples obtained from MCS (M = 1,000,000), the expected failure probability is calculated as 2.206×10^{-2}. The CoV of the failure probability is 2.174. Several key confidence intervals based on quantiles can also be determined, including the 95% confidence interval for the failure probability, given as $[2.231\times10^{-8}, 1.623\times10^{-1}]$, the 90% confidence interval of the failure probability, given as $[4.051\times10^{-7}, 1.102\times10^{-1}]$, and the 75% confidence interval of the failure probability,

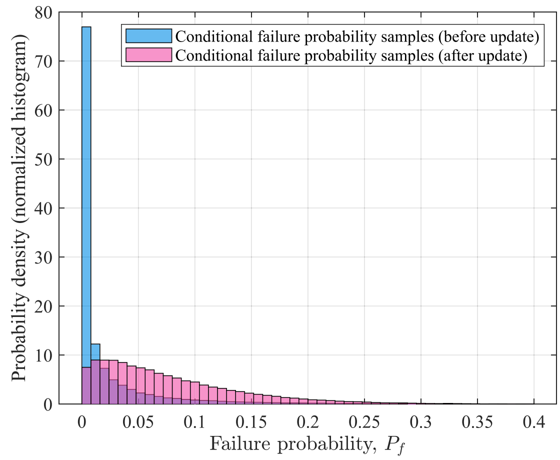


Fig. 7. Histogram of the failure probability before and after update.

given as $[1.523 \times 10^{-5}, 5.211 \times 10^{-2}]$. Furthermore, the PDF and CDF of the failure probability can be directly derived. In addition to the MCS method, Hong [64] employed the point-estimate method (PEM) to evaluate the expected failure probability, while Li et al. [21] utilized the distribution of the reliability index to determine the expected failure probability. To obtain quantile values of the failure probability, Zhao et al. [19] and Li et al. [21] utilized the distribution information of the reliability index to achieve this. While the mean, quantiles, or confidence intervals of the failure probability provide valuable insights for uncertainty-informed decision-making, they combine the effects of both aleatory and epistemic uncertainties, making it difficult to analyze the influence of epistemic uncertainty separately. Thus, it is crucial to obtain the probability distributions (PDF/CDF) of the failure probability, as this allows for the distinction between aleatory and epistemic uncertainties and enables a comprehensive quantification of the failure probability. Also, the PDF/CDF can offer the means to capture the effect of aleatory uncertainty, which is indexed with respect to the degree of epistemic uncertainty. To provide explicit PDF/CDF for the failure probability, Der Kiureghian and Ditlevsen [2] utilized the first-order approximation methodology. Additionally, Li et al. [21] proposed explicit probability distributions of the failure probability based on the method of moments. Since the models mentioned above incorporate a distribution of epistemic uncertainty, this suggests that they approach the problem from an internal perspective, keeping the effect of epistemic uncertainty separated from aleatory uncertainty.

After determining the distribution of the failure probability, its CoV and confidence interval can serve as indicators of the epistemic uncertainty associated with the failure probability. A large CoV or confidence interval in the failure probability distribution (as calculated from Fig. 6) signifies substantial epistemic uncertainty. It can be noticed that best-case behavior for failure probabilities for the confidence interval (as calculated from Fig. 6) is always nearly zero, making it less informative in such extreme cases. When this uncertainty is too wide for the purposes of analysis, further steps must be taken. To identify which input random variables have the greatest impact on failure probability P_f, global reliability sensitivity indices can be used [65,66]. After knowing the importance degree of the impact of each input uncertain parameters' epistemic uncertainty on the failure probability, actions should be taken to reduce the epistemic uncertainty of the most influential parameters.

Since the epistemic uncertainty is modeled within the framework of probability theory, it can be seamlessly integrated into Bayesian methods to reduce the epistemic uncertainty. When new measurements (denoted as $D_{\boldsymbol{X}}$) become available, the probability distributions of the uncertain parameters can be updated through Bayesian methods. This process incorporates the new information into the probabilistic model, refining prior distribution and reducing the associated epistemic uncertainty. Following the principle of Bayesian updating [67], the state of knowledge regarding the distribution parameters in Eq. (5) can be updated as follows:

$$f'_{\boldsymbol{\Theta}}(\boldsymbol{\theta}|D_{\boldsymbol{X}}) = kL(D_{\boldsymbol{X}}|\boldsymbol{\theta})f_{\boldsymbol{\Theta}}(\boldsymbol{\theta}), \tag{20}$$

where $f_{\boldsymbol{\Theta}}(\boldsymbol{\theta})$ represents the initial ("prior") PDF of distribution parameters $\boldsymbol{\theta}$, reflecting the current state of knowledge regarding $\boldsymbol{\theta}$; $f'_{\boldsymbol{\Theta}}(\boldsymbol{\theta}|D_{\boldsymbol{X}})$ denotes the updated ("posterior") PDF associated with the distribution parameters, which incorporates the newly acquired data measurements $D_{\boldsymbol{X}}$; $L(D_{\boldsymbol{X}}|\boldsymbol{\theta})$ represents the likelihood function of the data measurements; and k represents the evidence, which can be formulated as follows:

$$k = \left[\int_{-\infty}^{\infty} L(D_{\boldsymbol{X}}|\boldsymbol{\theta})f_{\boldsymbol{\Theta}}(\boldsymbol{\theta})\mathrm{d}\boldsymbol{\theta}\right]^{-1}. \tag{21}$$

For each dataset of measurements for the random variables $\boldsymbol{X}$, represented as $D_{\boldsymbol{X}_i}$, where $i = 1, 2, \ldots, N_d$, the likelihood function in Eqs. (20)–(21) can be formulated as

$$L(D_{\boldsymbol{X}}|\boldsymbol{\theta}) \propto \mathbb{P}\left(D_{\boldsymbol{X}_i}|\boldsymbol{\Theta} = \boldsymbol{\theta}\right). \tag{22}$$

If the data measurement samples $D_{\boldsymbol{X}_i}$ are statistically independent given $\boldsymbol{\Theta} = \boldsymbol{\theta}$, then the likelihood function in Eq. (22) can be written as

$$L(D_{\boldsymbol{X}}|\boldsymbol{\theta}) = \prod_{i=1}^{N} f_{\boldsymbol{X}|\boldsymbol{\Theta}}\left(D_{\boldsymbol{X}_i}\middle|\boldsymbol{\theta}\right). \tag{23}$$

Eq. (23) expresses the likelihood as a product over all individual data points, which reflects the assumption of statistical independence between them. On the other hand, the likelihood function $L(D_{\boldsymbol{X}}|\boldsymbol{\theta})$ quantifies how well the observed data $D_{\boldsymbol{X}}$ is explained by different values of the distribution parameters $\boldsymbol{\theta}$. In the context of Bayesian updating, it serves as a key component in adjusting the prior distribution to form the posterior distribution.

In the Bayesian updating framework, researchers frequently utilize established expert knowledge to formulate prior probability distributions. When these priors are grounded in robust theoretical foundations and supported by sufficient high-quality data, Bayesian updating plays a critical role in effectively reducing epistemic uncertainty. Then the failure probability can be updated using the posterior probability distribution of the distribution parameters $f'_{\boldsymbol{\Theta}}(\boldsymbol{\theta}|D_{\boldsymbol{X}})$ in place of its prior distribution $f_{\boldsymbol{\Theta}}(\boldsymbol{\theta})$. It needs to be discussed here that in this case, the distribution parameters $\boldsymbol{\Theta}$ are updated based on the measurements to update the failure probability. However, several studies updated the failure probability by directly updating the random variables $\boldsymbol{X}$, rather than updating the distribution parameters. Directly updating $\boldsymbol{X}$ may initially seem effective for enhancing accuracy or computational efficiency, but it gives rise to some problems. The most critical issue is that the resulting the updated failure probability becomes a deterministic value that conflates aleatoric uncertainty (inherent randomness) with epistemic uncertainty (knowledge gaps), thereby blending data-driven updates with prior beliefs. In the context of decision-making, however, it is essential to maintain a clear distinction and quantitatively characterize both aleatoric and epistemic uncertainties in the failure probability to ensure robust and reliable results.

To continue the numerical example discussed throughout this paper, assume that the data measurements of X_1 and X_2 are available, such that $D_{\boldsymbol{X}} = [45, 25; 40, 30; 47, 32; 42, 32; 44, 36]$. Based on Eqs. (5) and (20), the histogram of the failure probability considering the prior and posterior distributions of the distribution parameters obtained from Markov Chain Monte Carlo (MCMC) combined with MCS (M = 1,000,000) is shown in Fig. 7.

In the Bayesian framework, since inference is based on the posterior distribution, the interval is no longer a confidence interval but a credibility interval. The CoV and credibility interval of the failure

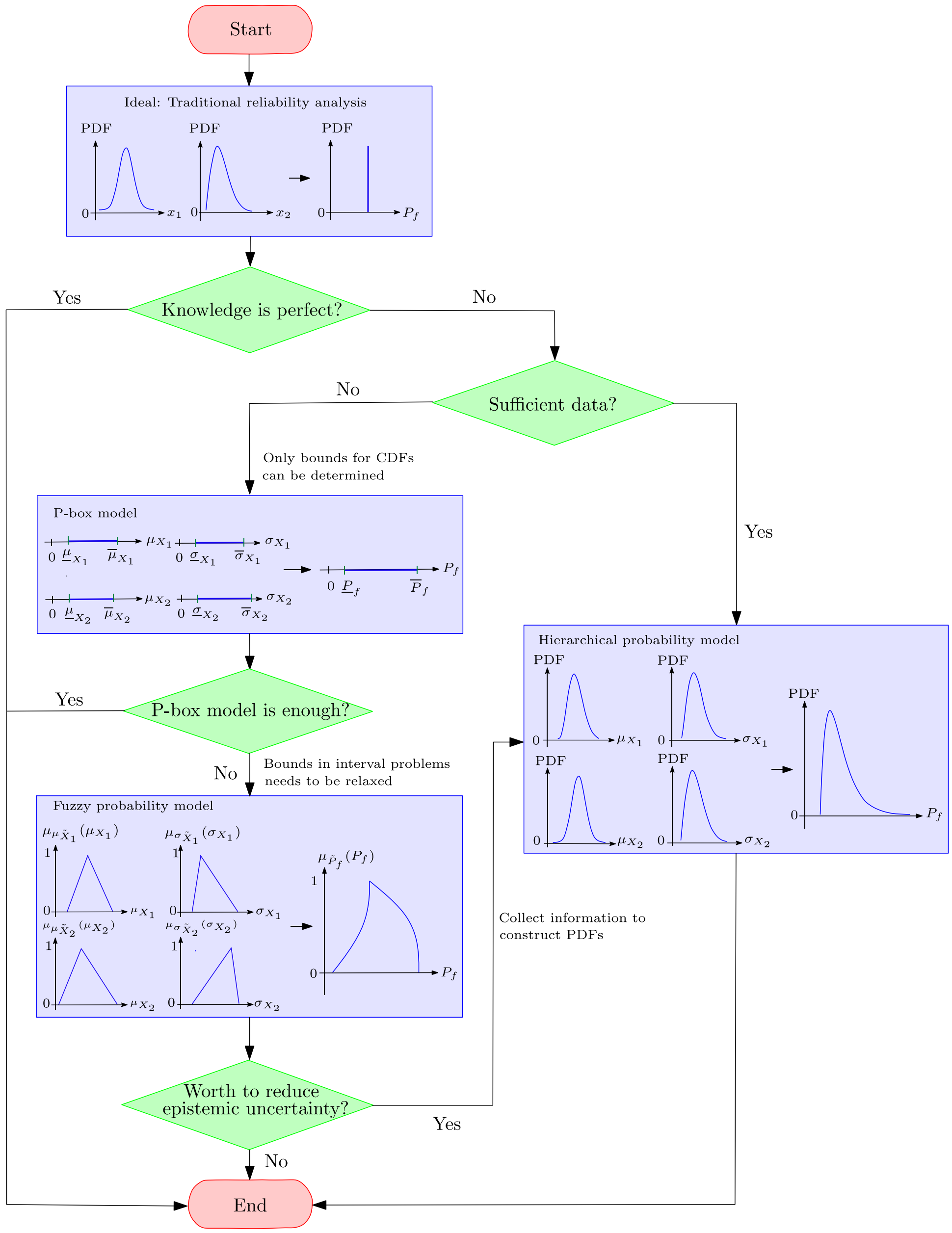


Fig. 8. Guidance on selecting the appropriate uncertainty modeling method.

probability, as illustrated in Fig. 7, serve as quantitative measures of epistemic uncertainty, reflecting the degree of imperfect knowledge. Following the update, the CoV decreased notably from 2.174 to 0.814, indicating a 63% reduction in relative epistemic uncertainty. This reduction indicates a shift from a higher to a lower degree of uncertainty regarding the failure probability, with lower CoV suggesting increased confidence in the estimates. Additionally, the credibility intervals for the failure probability have been updated as follows: the 95% credibility interval is $[3.720\times10^{-3}, 2.361\times10^{-1}]$; the 90% credibility interval is $[6.807\times10^{-3}, 2.013\times10^{-1}]$; and the 75% credibility interval is $[1.533\times10^{-2}, 1.513\times10^{-1}]$. Notably, the intervals for the same credibility level have narrowed after the update, which indicates that narrower credibility intervals reflect a lower degree of epistemic uncertainty, while wider intervals suggest a higher degree of epistemic uncertainty. As additional measurements become available, it is expected that both the CoV and credibility interval will continue to decrease, leading to progressively more accurate and reliable estimates. To achieve this, practitioners can systematically collect and incorporate new data, thereby reducing the epistemic uncertainty associated with failure probability. This iterative process involves continuously updating the probability distribution and its associated failure probability, refining estimates until the epistemic uncertainty meets predefined acceptability criteria. Achieving an acceptable level of uncertainty is crucial for strengthening the robustness of decision-making and enhancing the reliability of predictions. Ultimately, this approach fosters greater confidence in engineering analysis and risk assessments.

4. Practical notes

In structural reliability analysis, accurately accounting for both aleatory and epistemic uncertainties is essential. To achieve this, appropriate modeling approaches need to be carefully selected to effectively address these uncertainties, which depend on two perspectives. The first perspective focuses on the type and extent of available information, a common emphasis in quantification approaches within the literature. The second perspective, however, arises from the practical purpose of the analysis—namely, making specific decisions. In practice, the information required depends on the decision at hand. For instance, some decisions may rely on a rough estimate, such as an expected value or variance, which only necessitates a basic probabilistic analysis using the first two moments. In contrast, others may require a rigorous determination of upper probability bounds, which requires advanced methods such as imprecise probabilities or Chebyshev approaches [68]. From a practical standpoint, it is advisable to consider both perspectives when deciding on an appropriate modeling approach. As illustrated in Fig. 8, a practical engineering guide is invaluable for selecting suitable modeling methods and provides a comprehensive analysis of failure probabilities based on these two complementary perspectives. This guide begins with traditional reliability analysis, which focuses solely on aleatory uncertainty through probabilistic models. This initial step establishes a foundational understanding of structural reliability, assuming complete probabilistic knowledge of the random variables. However, this raises an important question: How much trust can we place in the probabilistic model itself? This question highlights the importance of epistemic uncertainty, which arises from incomplete or insufficient information and necessitates additional modeling techniques to ensure a comprehensive analysis.

The p-box model is a powerful tool for addressing epistemic uncertainty when information is scarce or incomplete. By establishing upper and lower bounds for unknown CDFs, it provides worst-case and best-case bounds on failure probabilities. This makes it particularly useful for safety-focused decisions, where the worst-case failure probability often suffices for ensuring acceptable safety margins. However, when problems require a focus on optimal design or sensitivity analysis and the intervals produced by the p-box model are too wide to support meaningful decisions, the p-box model may be inadequate. To address these limitations, the fuzzy probability model relaxes the rigid bounds of interval methods. This model evaluates the effects of perturbations in input intervals (e.g., slightly increasing or decreasing parameter ranges) on the failure probability, providing engineers with valuable insights into the overall sensitivity. By representing epistemic uncertainty as a collection of sets, the fuzzy probability model approaches the problem from the outside, focusing on the absolute worst-case and best-case behaviors while offering more flexible and informative bounds. After doing fuzzy probability analysis, one may decide to either perform Bayesian updating or not. If the upper bound for the failure probability is insensitive to the epistemic uncertainty and the designer only cares about the upper bound, then it has little practical benefit to reduce the epistemic uncertainty. Conversely, when the fuzzy failure probability exhibits significant sensitivity to the level of epistemic uncertainty, it becomes feasible — and often necessary — to collect additional data to construct the hierarchical probability model and further reduce epistemic uncertainty. The hierarchical probability model adopts a Bayesian approach to handle epistemic uncertainty from the inside when sufficient data is available to support probabilistic inference. It treats the uncertain distribution parameters as random variables, which transform the failure probability into a random variable with its own probability distribution. Each point on the probability distribution reflects aleatory uncertainty, corresponding to specific reliability analysis results. Unlike interval-based methods, which provide highly informative bounds, the hierarchical model's best-case behavior for the failure probability is always zero, making it less informative in such extreme cases. However, a key strength of the hierarchical model lies in its ability to reduce epistemic uncertainty during the Bayesian updating process as new information becomes available. In the ideal case, where epistemic uncertainty is fully resolved, the failure probability converges to a precise value.

It needs to be addressed that the concepts with interpretations from fuzzy set theory align closely with those from Bayesian subjective probabilities. In fuzzy set theory, one option for interpreting membership functions connects directly to belief quantification. Bayesian approaches, on the other hand, use prior distributions to quantify subjective probabilities. This offers two distinct yet complementary ways to express belief: through a membership function or a distribution function. While this distinction might seem non-rigorous at first glance, it is, in fact, rigorously supported. Evidence theory, particularly the Dempster-Shafer framework [69], further illustrates this connection. In evidence theory, membership functions are often interpreted as possibilities (a form of belief), which serve as upper bounds on probabilities for a given basic probability assignment. Notably, these assignments are typically subjective in nature, implying that possibilities effectively act as upper bounds on subjective probabilities. This perspective highlights the deeper coherence between Bayesian and fuzzy interpretations, demonstrating that the perceived contradictions between these concepts often arise from misunderstandings rather than fundamental conflicts. Moreover, the methods employed for modeling epistemic uncertainty — interval, fuzzy, and Bayesian inference — are compatible and can be used progressively or in an integrated manner. Each method provides a different resolution level, depending on the available information and analysis requirements. For example, when only coarse information is available, interval models may be sufficient. As more information becomes available or more refined modeling is needed, fuzzy and Bayesian methods can be incorporated. This compatibility ensures a smooth transition between modeling levels, supports flexible adaptation to different scenarios, and preserves the probabilistic foundation required for aleatory uncertainty. Therefore, integrating these approaches within a layered structure not only enhances the interpretability and rigor of reliability analyses but also enables analysts to maintain consistency and coherence across uncertainty modeling frameworks.

In conclusion, Fig. 8 offers a systematic approach that allows analysts to address both aleatory and epistemic uncertainties effectively while providing engineers with valuable guidance on applying established modeling techniques in practical scenarios. By integrating the strengths of various uncertainty modeling methods, engineers can choose the most appropriate approach depending on the specific goals of the problem—whether it involves ensuring safety, optimizing structural design, or refining reliability estimates. It should be emphasized, however, that the flowchart is intended to serve as a general framework and not as a rigid procedure. In practice, variations and deviations from the depicted process are often necessary.

When uncertainty remains about which method to use — especially for critical decisions — it is prudent to apply multiple approaches and compare the resulting decisions. Consistent conclusions across methods indicate robust decisions; otherwise, further information collection may be necessary to refine the analysis. On top of that, incorporating sensitivity analysis across these methods is also essential to identify critical parameters that influence failure probability, thereby guiding efficient

Table 2
Probabilistic information for the random variables.

Parameter	Distribution	Mean value	CoV
B	Deterministic	10 m	–
D	Gaussian (truncated)	1 m	0.15
γ	Lognormal	20 kN/m^3	0.1
c	Lognormal	20 kPa	0.25
ϕ	Beta	Range: [0, 45]°, $\mu_\phi = 30°$	≈0.1

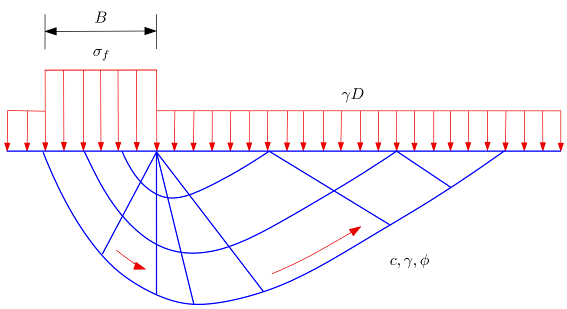


Fig. 9. Strip footing.

data collection and informed decision-making. Therefore, this framework addresses both theoretical quantification and decision-oriented analysis, supporting robust decision-making even in the presence of significant uncertainty. Using various modeling approaches and incorporating sensitivity analysis, engineers can achieve reliable and goal-oriented results in structural reliability analysis.

5. Application example

This section presents a classical strip footing problem from Sudret (2014) [70] to illustrate the application of each method investigated in this study. The strip footing has a width of $B = 10$ m and is embedded at a depth D, as illustrated in Fig. 9. It is assumed that groundwater is located far below the surface, and the soil is homogeneous with cohesion c, friction angle ϕ and unit weight γ. The limit-state function is defined as the ultimate bearing capacity not exceeding its design value [71]:

$$g(\boldsymbol{X}) = q_u(\boldsymbol{X}) - q_{des} = cN_c + \gamma D N_q + \frac{1}{2} B\gamma N_\gamma - q_{des}, \tag{24}$$

where the bearing capacity factors are given as follows:

$$\begin{aligned} N_q &= e^{\pi \tan\phi} \tan^2\left(\frac{\pi}{4} + \frac{\phi}{2}\right), \\ N_c &= (N_q - 1)\cot\phi, \\ N_\gamma &= 2(N_q - 1)\tan\phi. \end{aligned} \tag{25}$$

The depth D, unit weight γ, cohesion c, and friction angle ϕ are treated as independent random variables with properties listed in Table 2. The design value q_{des} is obtained by dividing the ultimate bearing capacity by the safety factor (SF), i.e., $q_{des} = \overline{q_u}/\text{SF}$. The value of $\overline{q_u}$ is determined using the mean values of the parameters listed in Table 2. With a safety factor of 2, the resulting design value is q_{des}= 2.78 MPa.

Using the probability information provided in Table 2, a reliability analysis is performed considering only aleatory uncertainty. The failure probability, defined in Eq. (1), is estimated via MCS considering 10^6 samples as $\hat{P}_f = 3.9345 \times 10^{-2}$. Due to limited information, there is significant epistemic uncertainty in estimating the mean values of γ and c (i.e., μ_γ and μ_c), as well as the distribution parameters of ϕ (i.e., α_ϕ and β_ϕ). It is assumed that the interval values of μ_γ, μ_c, α_ϕ and β_ϕ are given as follows: $\mu_\gamma \in [19, 21]$ kN/m^3, $\mu_c \in [18, 22]$ kPa, $\alpha_\phi \in [29, 35]$, and $\beta_\phi \in [13, 19]$. Given the known distribution families (Table 2), parametric p-boxes are used to model the CDFs of γ, c, and ϕ, as given in Fig. 10.

Table 3
Probabilistic information for prior distribution of the distribution parameters.

Parameter	Distribution	Mean value	CoV
μ_γ	Lognormal	20 kN/m^3	0.075
μ_c	Lognormal	20 kPa	0.075
α_ϕ	Lognormal	32	0.075
β_ϕ	Lognormal	16	0.075

Based on the parametric p-box of γ, c, and ϕ shown in Fig. 10, the bounds of the failure probability are estimated by decoupling the treatment of aleatory and interval uncertainties. This is achieved through an iterative boundary search employing particle swarm optimization (PSO) [72]. For each parameter realization, the corresponding failure probability is estimated using MCS with 10^6 samples. Then the lower and upper bounds of the failure probability are given as:

$$\underline{\hat{P}}_f = \mathbb{P}\left(\max_{X_i \in [\underline{X}_i, \overline{X}_i]} g(\boldsymbol{X}) \leq 0\right) = 2.769 \times 10^{-3}, \tag{26}$$

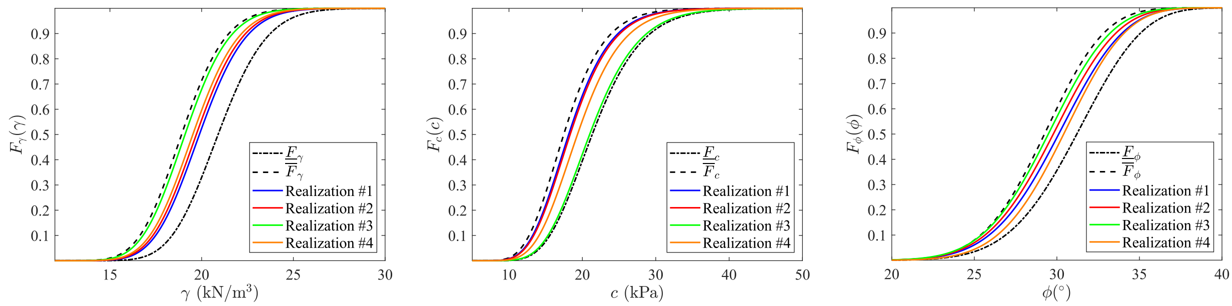
$$\overline{\hat{P}}_f = \mathbb{P}\left(\min_{X_i \in [\underline{X}_i, \overline{X}_i]} g(\boldsymbol{X}) \leq 0\right) = 2.243 \times 10^{-1}. \tag{27}$$

As shown in Eqs. (26)–(27), the bounds for the failure probability span two orders of magnitude, which is too wide for general analysis to be meaningful. More analysis of the epistemic uncertainty is needed to determine whether additional data collection and which types of data are needed and beneficial. Before collecting additional data, the fuzzy probability is used to identify whether the epistemic uncertainty worth to be reduced based on the overall trends in the failure probability across the uncertainty landscape. Thus, the triangular membership functions are used to model μ_γ, μ_c, α_ϕ, and β_ϕ, which are given in Fig. 11.

Referring to the fuzzy probability model with the triangular membership functions as shown in Fig. 11, the fuzzy failure probability can be obtained based on Eqs. (17)–(18), which is shown in Fig. 12. As shown in Fig. 12, the failure probability corresponding to a membership value of 1 is $\hat{P}_f = 3.9172 \times 10^{-2}$, which closely matches the result obtained without accounting for epistemic uncertainty. When the membership value is 0, the failure probability lies within the range $[\underline{\hat{P}}_f = 2.866 \times 10^{-3}, \overline{\hat{P}}_f = 2.226 \times 10^{-1}]$, which is consistent with the bounds derived from the parametric p-box model. The overall shape of the membership function clearly indicates that the failure probability is highly sensitive to the level of epistemic uncertainty. This sensitivity suggests that collecting additional data could significantly reduce epistemic uncertainty and enhance the reliability of decision-making. Motivated by this observation, the hierarchical probability model is adopted to incorporate the epistemic uncertainty in a Bayesian framework. Accordingly, the distributional assumptions for the distribution parameters μ_γ, μ_c, α_ϕ, and β_ϕ are specified as shown in Table 3 in terms of their respective prior distributions.

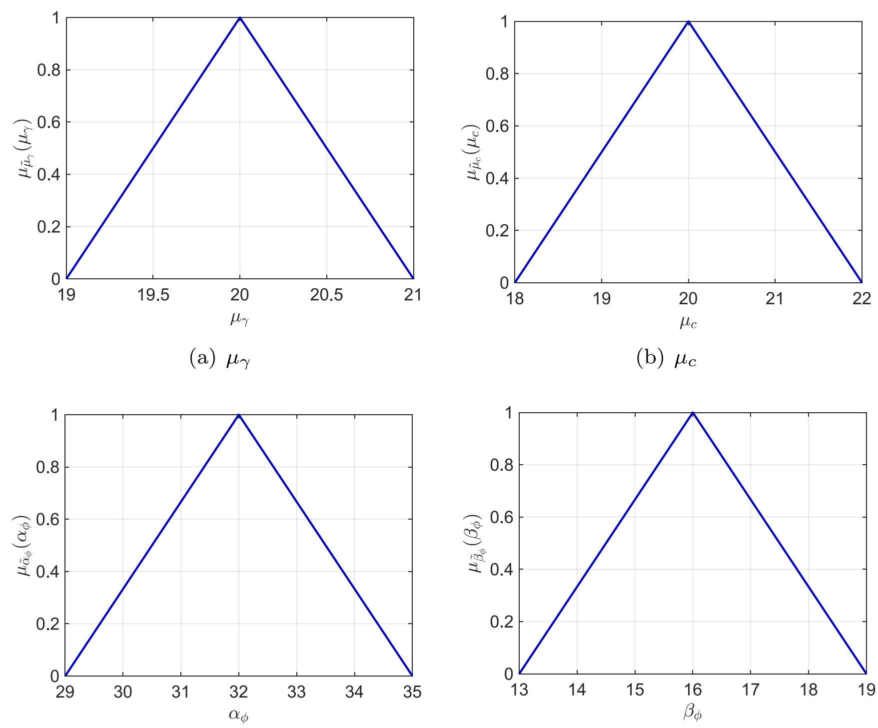
Using the probabilistic information of the distribution parameters provided in Table 3 and following Eq. (2), samples of conditional failure probability are generated via a double-loop MCS (10^6 times), as illustrated in Fig. 13. Based on the conditional failure probability samples shown in Fig. 13, the expected failure probability P_F is estimated as 5.345×10^{-2}, with the CoV of the conditional failure probability being 0.750. Several key quantile-based confidence intervals are also derived, including the 95% confidence interval is $[7.441 \times 10^{-3}, 1.585 \times 10^{-1}]$, the 90% confidence interval is $[1.019 \times 10^{-2}, 1.317 \times 10^{-1}]$, and the 75% confidence interval is $[1.631 \times 10^{-2}, 9.692 \times 10^{-2}]$.

Furthermore, additional soil property measurements are taken from Raviteja et al. [73] of soil weight γ, cohesion c, and friction angle ϕ, given as $[D_\gamma, D_c, D_\phi] = [14.1, 23, 24; 17.6, 29, 39; 15.9, 28, 32; 12.5, 30, 37; 13.2, 18, 29.2]$. These measurements are used to update the probability distributions of the distribution parameters through Bayesian inference. Subsequently, based on Eqs. (2) and (20), the histogram of the prior and



(a) Soil weight γ (b) Cohesion c (c) Friction angle ϕ°

Fig. 10. Parametric p-box of γ, c, and ϕ.



(a) μ_γ (b) μ_c (c) α_ϕ (d) β_ϕ

Fig. 11. Membership functions of μ_γ, μ_c, α_ϕ, and β_ϕ.

updated failure probability, reflecting both prior and posterior distributions of the distribution parameters obtained from MCMC combined with MCS (M = 1,000,000) is shown in Fig. 14.

As shown in Fig. 14, the CoV and the credibility intervals of the failure probability serve as quantitative indicators of epistemic uncertainty, reflecting the degree of imperfect knowledge in the system. Following the Bayesian updating, the CoV of the failure probability is reduced from 0.750 to 0.383, indicating a 49% reduction in relative epistemic uncertainty. The credibility intervals for the failure probability have been updated as follows: the 95% credibility interval is $[1.074\times 10^{-1}, 5.471\times 10^{-1}]$; the 90% credibility interval is $[1.295\times 10^{-1}, 5.034\times 10^{-1}]$; and the 75% credibility interval is $[1.694\times 10^{-1}, 4.343\times 10^{-1}]$. As additional measurements become available, it is expected that both the CoV and credibility interval will decrease, leading to progressively more accurate and reliable estimates. This refined uncertainty characterization has important implications for rational decision-making in geotechnical design under both aleatory and epistemic uncertainties.

6. Summary and conclusions

This study provides a comprehensive investigation into the uncertainty modeling techniques for input uncertain variables with aleatory and epistemic uncertainties in structural reliability analysis from an engineering perspective. While traditional probabilistic methods offer a foundational understanding of the failure probability by focusing exclusively on aleatory uncertainty, they often prove inadequate in scenarios characterized by incomplete, insufficient, imperfect, or imprecise data. To overcome these limitations, we explored three widely used models—p-boxes, fuzzy probability, and hierarchical probability—which provide

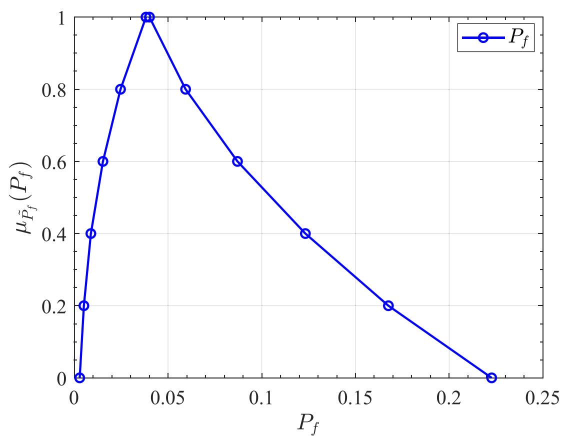


Fig. 12. Membership function of the failure probability.

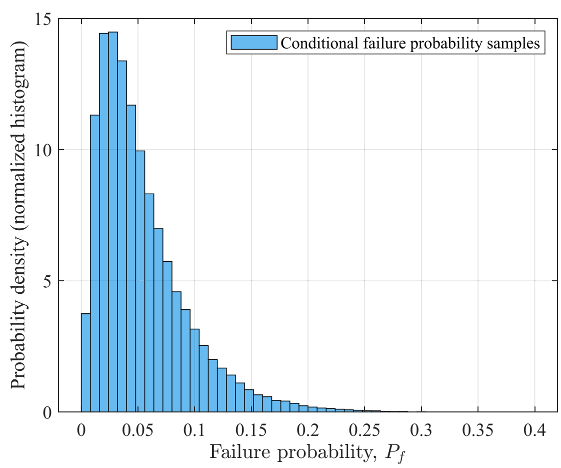


Fig. 13. Histogram of the failure probability.

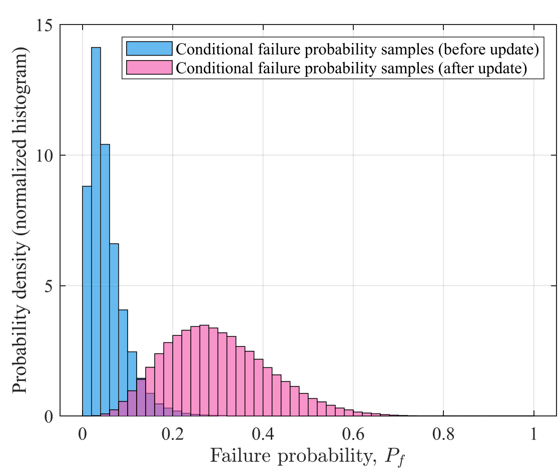


Fig. 14. Histogram of the failure probability before and after update.

robust frameworks for integrating epistemic uncertainty together with aleatory analysis.

The p-box model offers efficient and conservative bounds for the failure probability, making it particularly suited for safety-critical scenarios. However, its rigid intervals may limit its applicability to structures requiring detailed sensitivity or optimization insights. The fuzzy probability model extends this capability by capturing the effects of parameter perturbations, yielding flexible and informative bounds that support a broader understanding of uncertainty propagation and reliability analysis. Complementing this, the hierarchical probability model employs the Bayesian theory to reduce the epistemic uncertainty. By representing uncertainties probabilistically and incorporating new data iteratively, it enables the failure probability to converge to a precise value under ideal conditions, making it highly applicable in evolving engineering contexts. Notably, fuzzy set theory and Bayesian probabilities share a coherent foundation, as both membership functions and prior distributions represent belief. Additionally, we underscore the pivotal role of sensitivity analysis in uncertainty modeling. By identifying critical parameters that significantly influence the failure probability, sensitivity analysis guides targeted data collection to reduce epistemic uncertainty and improve reliability predictions. However, we note that global sensitivity analysis methods, particularly those integrating aleatory and epistemic uncertainties, face ongoing methodological challenges. Addressing these challenges is essential for improving the precision and applicability of such analyses in practical engineering scenarios.

In conclusion, this study thoroughly investigates the strengths and limitations of existing frameworks for modeling input uncertain variables with aleatory and epistemic uncertainties in structural reliability analysis. It offers engineers a comprehensive overview, enabling them to select the most suitable approach based on the specific requirements of their problems, make more informed decisions, and achieve safer and more efficient designs. The main recommendation from this study is not to calculate the traditional reliability analysis that only considers aleatory uncertainty, especially when epistemic uncertainty is present. Additionally, computationally expensive hierarchical Bayesian models should be used when simpler methods, such as p-box models are insufficient for decision-making. Future research should focus on refining these models, enhancing sensitivity analysis techniques, and advancing their application to increasingly complex engineering problems, ensuring a seamless transition from theoretical development to practical implementation.

CRediT authorship contribution statement

Pei-Pei Li: Writing – review & editing, Writing – original draft, Software, Methodology, Investigation, Funding acquisition, Conceptualization. **Marcos A. Valdebenito:** Writing – review & editing, Methodology, Investigation, Conceptualization. **Chao Dang:** Writing – review & editing, Software, Investigation, Funding acquisition. **Michael Beer:** Writing – review & editing, Methodology. **Matthias G.R. Faes:** Writing – review & editing, Supervision, Methodology, Investigation, Funding acquisition, Conceptualization.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Appendix. List of symbols

B	Width of strip footing (m)
C_i	CDF value of auxiliary uniform variable mapped to X_i
D	Depth of strip footing (m)
$D_{\boldsymbol{C}}$	Domain of $\boldsymbol{C}$, i.e., $[0,1]^K$
D_{X_i}	Domain of random variable X_i
$D_{\boldsymbol{\theta}}$	Domain of $\boldsymbol{\theta}$
$D_{\boldsymbol{X}}$	Observed data associated with $\boldsymbol{X}$
$F_{X_i}(x_i)$	Cumulative distribution function of X_i
$F_{X_i}^{-1}$	Inverse of the CDF of X_i
$\underline{F}_{X_i}(x_i), \overline{F}_{X_i}(x_i)$	Lower and upper bounds of the CDF for X_i
$\underline{F}_{X_i}^{-1}, \overline{F}_{X_i}^{-1}$	Inverse of lower/upper bound of the CDF for X_i
$\underline{F}_{X_i}^{\alpha}(x_i), \overline{F}_{X_i}^{\alpha}(x_i)$	Lower and upper bounds of CDF at α-level for X_i
$\hat{P}_f$	Estimated failure probability
$\hat{\underline{P}}_f, \hat{\overline{P}}_f$	Estimated lower and upper bounds of failure probability
$L(D_{\boldsymbol{X}} \mid \boldsymbol{\theta})$	Likelihood of data $D_{\boldsymbol{X}}$ given distribution parameters $\boldsymbol{\theta}$
N_c, N_q, N_γ	Bearing capacity factors (dimensionless)
N_d	Number of data samples
P_F	Expected failure probability over parameter uncertainty
P_f	Failure probability, i.e., probability that $g(\boldsymbol{X}) \le 0$
$P_f(\boldsymbol{\theta})$	Conditional failure probability for given parameter vector $\boldsymbol{\theta}$
$P_{f,\alpha}^I$	Interval-valued failure probability at α-cut
$\underline{P}_f, \overline{P}_f$	Lower and upper bounds of failure probability
$\underline{P}_f^{\alpha}, \overline{P}_f^{\alpha}$	Lower and upper bounds of failure probability at α-cut
$\boldsymbol{\Phi}(\cdot)$	Standard normal CDF
α	Membership level in $(0,1]$ for α-cut
β	Reliability index
$\beta(\boldsymbol{\theta})$	Conditional reliability index for given parameter vector $\boldsymbol{\theta}$
$\boldsymbol{C}$	Vector of auxiliary uniform variables on $[0,1]^K$
$\boldsymbol{X}$	Random input vector with components X_i
$\boldsymbol{\theta}$	Distribution parameter vector
$\boldsymbol{\theta}_{\alpha}^I$	α-cut interval for all uncertain parameters
c	Soil cohesion (kPa)
$f_{\boldsymbol{X}}(\boldsymbol{x})$	Joint PDF of $\boldsymbol{X}$
$f_{\boldsymbol{X}}(\boldsymbol{x} \mid \boldsymbol{\theta})$	Conditional PDF of $\boldsymbol{X}$ given distribution parameters $\boldsymbol{\theta}$
$f_{\boldsymbol{\Theta}}(\boldsymbol{\theta})$	Prior PDF of distribution parameters $\boldsymbol{\theta}$
$f'_{\boldsymbol{\Theta}}(\boldsymbol{\theta} \mid D_{\boldsymbol{X}})$	Posterior PDF of distribution parameters given data $D_{\boldsymbol{X}}$
$g(\boldsymbol{X})$	Limit state function (e.g., $g(\boldsymbol{X}) = X_1 - X_2$)
$\underline{g}(\cdot), \overline{g}(\cdot)$	Lower and upper bounds of limit state function
γ	Unit weight of soil (kN/m^3)
k	Bayesian model evidence (normalization constant)
$\mathbb{F}$	Fuzzy set of admissible CDF values
$\mathbb{P}(\cdot)$	Probability operator
$\underline{\mathbb{P}}(\cdot), \overline{\mathbb{P}}(\cdot)$	Lower and upper probability operator
μ_{X_i}	Mean of random variable X_i
$\mu_{\tilde{\theta}_{i,l}}(\theta_{i,l})$	Membership function of $\theta_{i,l}$
$\mu(F_{X_i}(x_i \mid \boldsymbol{\theta}_i))$	Membership degree of a specific CDF $F_{X_i}(x_i \mid \boldsymbol{\theta}_i)$
ϕ	Soil internal friction angle (degrees)
q_{des}	Design bearing pressure (kPa)
q_u	Ultimate bearing capacity of soil (kPa)
σ_{X_i}	Standard deviation of X_i
$\tilde{F}_{X_i}(x_i \mid \boldsymbol{\theta}_i)$	Fuzzy CDF of X_i conditioned on $\boldsymbol{\theta}_i$
$\tilde{F}_{X_i}^{\alpha}(x_i \mid \boldsymbol{\theta}_i)$	Fuzzy CDF at level α
$\tilde{\theta}_{i,l}$	Fuzzy set of the lth distribution parameter of X_i
SF	Safety factor
$\theta_{i,l}$	lth distribution parameter of X_i
$\theta_{i,l,\mu}$	Most probable value of $\theta_{i,l}$ (membership = 1)
$\theta_{i,l,\alpha}^I$	α-cut interval for $\theta_{i,l}$
$\underline{\mu}_{X_i}, \overline{\mu}_{X_i}$	Upper and lower bounds on μ_{X_i}
$\underline{\sigma}_{X_i}, \overline{\sigma}_{X_i}$	Upper and lower bounds on σ_{X_i}
$\underline{\theta}_{i,l}, \overline{\theta}_{i,l}$	Lower and upper bound of $\theta_{i,l}$ (membership = 0)
$\underline{\theta}_{i,l,\alpha}, \overline{\theta}_{i,l,\alpha}$	Lower and upper bounds of α-cut for $\theta_{i,l}$
$\underline{X}_i, \overline{X}_i$	Lower and upper of X_i

Data availability

Data will be made available on request.

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