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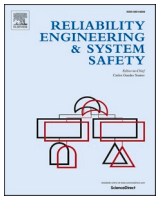

Asymptotic subset simulation: An efficient extrapolation tool for small probabilities approximation

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ABSTRACT

This study bridges the concepts of subset simulation with asymptotic approximation theory in multinormal integrals for the estimation of small probabilities. To meet this aim, for a sequence of scaled limit state functions (LSFs) with failure probabilities higher than the original LSF, it is found that the proposed asymptotic approximation and subset simulation can be applied within the same framework, and only a few steps of subset simulation could be sufficient to approximate small failure probabilities using extrapolation. The analogy of the formulation of the second-order reliability method (SORM) with the proposed concept is studied and, considering sequential sampling as a search algorithm, shown that the information obtained from a few steps of the design point search process could be enough to approximate the total failure probability of a problem. Solving intricate nonlinear and high-dimensional problems confirms the efficiency and robustness of the proposed framework for reliability analysis of real-world engineering problems with small probabilities.

1. Introduction

Probability analysis, uncertainty management, and decision making under uncertainty are integral to diverse fields like engineering [1–3], medicine [4], and computer science [5,6]. Structural reliability theory presents a mathematical tool for handling these topics, which clarifies the importance of this theory as a multidisciplinary concept with a wide range of applications. Reliability analysis in essence requires solving a probability integral involving a multi-variable function g (i.e., the model), known as performance function, with random variables $\boldsymbol{X}$ (i.e., uncertain parameters):

$$P_E = \int_E f_D(\boldsymbol{x})\mathrm{d}\boldsymbol{x}, \tag{1}$$

where E is the event of interest (e.g., failure as $E = \{g(\boldsymbol{x}) \leq 0\}$) and f_D is the joint probability density function (PDF) of the set D grouping the random variables involved [7]. In this context, extremely low probabilities, complex function topology in probability space, high-dimensional problems with numerous and different types of random variables, and costly or time-consuming modeling are known as the main challenges of reliability analysis [8–10]. To address the mentioned challenges, efficient reliability methods (such as first and second-order reliability methods (FORM and SORM) [11–13], Monte Carlo simulation [14–16], importance sampling [17–19], directional sampling [20], line sampling [21–23], subset simulation [24–26], and probability density evolution methods [27,28], etc.) have been developed for solving probability integrals. Broadly speaking, reliability methods can be broadly classified as approximate, simulation-based, or surrogate-based [29–32].

The approximation approaches mainly relied on first finding the important failure region of the problems (usually by using an optimization algorithm for locating design points) and providing an approximation of failure probability by employing some assumptions about the limit state function (LSF) of the problem [16]. The FORM and SORM [11,13] are popular reliability approaches in this category. Having information about the LSF of the problem (e.g., the number of important failure domains of the problem and moderate nonlinearity of LSF around design points) would lead to an acceptable approximation of the failure probability with a reasonable number of function calls under certain conditions [33].

Simulation methods estimate failure probabilities based on random

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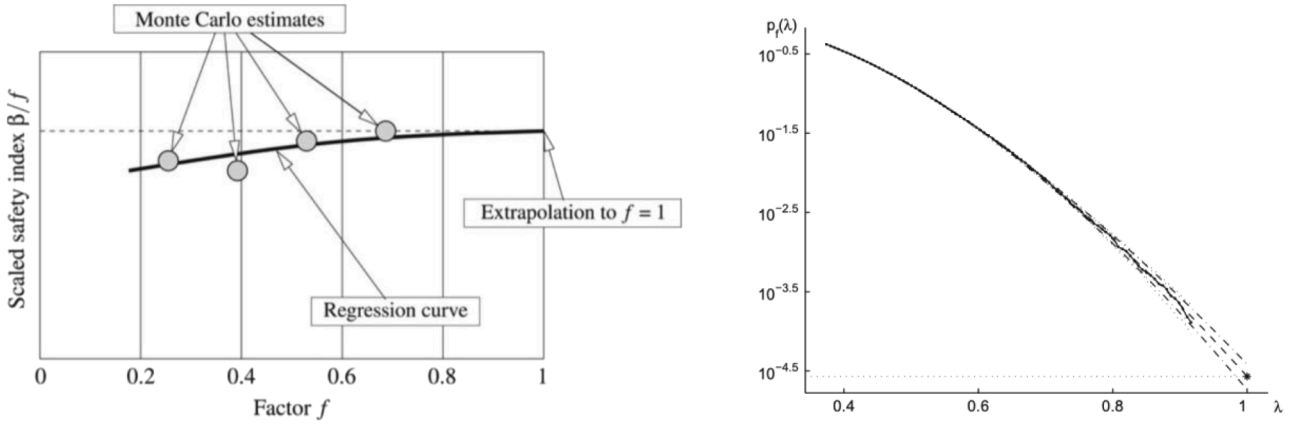

Fig. 1. Left: The asymptotic sampling [45] and, Right: Extreme values theory [50] for small probability estimation.

sampling. Monte Carlo simulation (MCS) [34] is a well-known approach in this category which approximates the failure probability with no assumption about the LSF and therefore always reports proper results with a certain confidence interval. For such a performance, MCS is required to generate samples in physical space according to the PDF of random variables and evaluate the performance of the system for each sample which requires a large number of function calls for the estimation of small probabilities. Because of this issue, the application of MCS for reliability analysis of many real-world engineering problems (e.g., those involving finite element-based models) is very challenging. This issue led to the development of other sampling-based methods that mainly work based on the idea of MCS but use some assumptions in analysis to reduce the burden of function calls for probability estimation [17,24,35]. The traditional subset simulation and sequential importance sampling are popular approaches in this category [24,36]. The main assumption behind this method is that the topology of the performance function allows the generating of conditional samples that approach important failure regions of the problem [37] (noting the point that recently, some new versions of these methods have been developed that work without such assumptions [16,25]). In the Line sampling method, the main assumption is that (the direction of) the important failure regions is available and this approach captures the curvature of the LSF (at important failure regions). Recently, the soft Monte Carlo simulation [15] has been developed as an extension of classical line sampling.

While the proposed approaches suitably reduce the function call of MCS, their number of full function calls for the estimation of (small) failure probabilities might still be too large for real-world applications. On top, if the assumption behind the method does not fit the structure of the in-hand problem, the same as the approximation methods, they may report wrong results [38]. For instance, in contrast with the assumptions behind subset simulation, MCMC may fail to transmit samples toward proper important failure regions. Similarly, despite interesting attempts made to address the drawbacks of line sampling [15,39], specifically in the case of high dimensional problems, the employed important direction may not highlight the proper important failure region of the problem.

In the case of surrogate based methods [40,41], the focus of the most of approaches is to interpolate the performance function for a set of input samples $\boldsymbol{X}$ [42–44]. Nonetheless, there is also a class of methods that directly approximate the tail of the PDF/CDF function of g. The proposed approaches are generally developed for the estimation of very small probabilities and mainly work based on the asymptotic behavior of failure probability concerning the change in statistical parameters of random variables or performance function g. The asymptotic sampling [45–48] and the method developed based on extreme values theory [49–51] can be considered as the main cases of this approach. While the asymptotic sampling focuses on the behavior of reliability indices of a series of scaled LSFs, extreme value theory suggests an exponential function for approximating the failure probability (See Fig. 1).

Considering the proposed implementations, the main contribution of this study is the connection of the concept of sequential simulation approaches to the extreme values theory. This paves the way for consolidating some numerical and approximation methods within a unified framework and provides the opportunity to use the capabilities of both categories in probability estimation.

In the next section, we present the asymptotic subset simulation framework by explaining the analogy of subset simulation with the asymptotic approximation in multinormal probability integrals. Then, the formulation of the SORM is compared with the proposed framework, and new extrapolation functions that are specifically suited for the approximation of small probabilities are suggested. Section 2.3. presents the generalized form of the proposed approach and then, the statistical properties of the estimations are investigated in Sections 2.4. The proposed framework is verified in Section 3 and the conclusions are reported in Section 4.

2. Asymptotic subset simulation

2.1. Concept

The probability integral of Eq. (1) can be mapped to standard normal space using isoprobabilistic transformations [13]. For such an integral with D random variables, consider a problem with a small failure probability P_f which $g(\boldsymbol{u})$ and Ω represent its LSF and failure domain in standard normal space (i.e., if $\boldsymbol{u} \in \Omega$, then $g \leq 0$). Then, the failure probability of this system can be presented as:

$$P_{f(\Omega)} = \frac{1}{\sqrt[D]{2\pi}} \int_{\Omega=\{g(\boldsymbol{u})\leq 0\}} \exp\left\{-\frac{1}{2}|\boldsymbol{u}|^2\right\} \mathrm{d}\boldsymbol{u}, \tag{2}$$

which is often difficult to evaluate, e.g., when the function $g(\boldsymbol{u})$ is highly nonlinear, the number of random variables is too large, especially when the failure probability is very small (see Section 1 for more details). To solve the proposed problem efficiently, by applying a change in the parameters of the random variables or function g, we define a weakened system with LSF of $\widehat{g}$ and the failure domain of $\Theta = \{\widehat{g}(\boldsymbol{u}) \leq 0\}$ with a failure probability as follows:

$$P_{f(\Theta)} = \frac{1}{\sqrt[D]{2\pi}} \int_{\Theta=\{\widehat{g}(\boldsymbol{u})\leq 0\}} \exp\left\{-\frac{1}{2}|\boldsymbol{u}|^2\right\} \mathrm{d}\boldsymbol{u}, \tag{3}$$

that is aimed to have two main specifications:

1) The target probability $P_{f(\Theta)}$ be suitably higher than the original probability (i.e., $P_{f(\Theta)} \gg P_{f(\Omega)}$).

Table 1
Some alternative approaches for weakening a system and increasing the failure probability.

	Scaled performance function $\widehat{g}(\boldsymbol{u})$	Description
1	$g(\lambda \boldsymbol{u})$	Increasing the variance of random variables
2	$g(\boldsymbol{u}) - \mathrm{T}(\lambda)$	Reducing the performance function based on λ Example: $\widehat{g}(u) = g(\boldsymbol{u}) - \gamma(1-\lambda)$ where γ=cte [50]
3	$R(\boldsymbol{u}\lambda) - Q(\boldsymbol{u})$	Decreasing the resistance of the systems (or/and) increasing the load parameter
4	$R(\boldsymbol{u}\lambda) - Q(\boldsymbol{u}\lambda^{-1})$	
5	$\mathrm{T}(\lambda)R(\boldsymbol{u}) - Q(\boldsymbol{u})$	
6	$R(\boldsymbol{u}) - \mathrm{T}(\lambda^{-1})Q(\boldsymbol{u})$	
7	$\mathrm{T}(\lambda)R(\boldsymbol{u}) - \mathrm{T}(\lambda^{-1})Q(\boldsymbol{u})$	

2) The failure domain $\Theta = \{\widehat{g}(\boldsymbol{u}) \le 0\}$ reflects a scaled version of the original failure domain $\Omega = \{g(\boldsymbol{u}) \le 0\}$, while including all important failure regions of the original problem (i.e., $\Omega \in \Theta$).

To meet this dual aim, consider $Q(\boldsymbol{u})$ and $R(\boldsymbol{u})$ as load and resistance parameters of the problem respectively. Then, one may use a scale parameter λ to form a new LSF as $\widehat{g}(\boldsymbol{u}) = g(\boldsymbol{u};\lambda) = R(\boldsymbol{u};\lambda) - Q(\boldsymbol{u};\lambda)$ with the failure domain of $\Theta = \{\widehat{g}(\boldsymbol{u}) = g(\boldsymbol{u};\lambda) \le 0\}$. For example, based on the specification of the in-hand problem, one of the forms of the performance functions in Table 1 may be used to obtain a weakened system in which the scaled function meets the original LSF when $\lambda = 1$.

Note that the scaling approach is not limited to the cases presented in Table 1. In addition, the scale parameter may be used to change the statistical parameters of the random variables to obtain a weakened system.

Now, without losing generality, the failure probability of the system can be reformulated using the control variate technique as follows [8]:

$$P_{f(\Omega)} = \ell P_{f(\Theta)} + \frac{1}{\sqrt[p]{2\pi}}\left(\int_{\Omega} \exp\left\{-\frac{1}{2}|\boldsymbol{u}|^2\right\} d\boldsymbol{u} - \ell \int_{\Theta} \exp\left\{-\frac{1}{2}|\boldsymbol{u}|^2\right\} d\boldsymbol{u}\right), \tag{4}$$

in which ℓ is the control parameter. As shown in Table 1, there are several alternative approaches for scaling the failure domain. Nonetheless, in this section, we focus on a specific case in which the desired failure domain is obtained from the LSF of $\widehat{g}(\mathrm{u}) = g(u;\lambda) = g(\lambda^{-1}u)$ where $\lambda < 1$. This is performed to stay close to the concept presented in Ref [13] with asymptotic estimations (See also Proposition 1 for more details). Mathematically, this approach is equal to the change in variance of random variables in the original u-space, and therefore, Eq. (4) can be reformulated as follows:

$$P_{f(\Omega)} = \ell P_{f(\Theta)} + \frac{1}{\sqrt[p]{2\pi}} \int_{\Omega} \left(\exp\left\{-\frac{1}{2}|\boldsymbol{u}|^2\right\} - \lambda^p \ell \cdot \exp\left\{-\frac{\lambda^2}{2}|\boldsymbol{u}|^2\right\}\right) \mathrm{d}\boldsymbol{u}. \tag{5}$$

In Eq. (5), the first term on the right-hand side of the equation (the $P_{f(\Theta)}$ with the LSF of $\widehat{g}(\mathbf{u}) = g(\boldsymbol{u};\lambda)$) corresponds to a high-probability event and is therefore easy to estimate. At the same time, the estimation of the integral which represents the difference of PDFs may be more challenging. The geometrical representation of the proposed implementations is presented in Fig. 2 for the Metaball function (See [37] and Section 3.1 for more information).

Consider Ω and Θ respectively as the failure domains of the original and fictitious LSFs. By increasing the scale parameter λ from λ_1 to 1 (e.g., $\boldsymbol{\lambda} = [\lambda_1, \lambda_2, \ldots, \lambda_{S-1}, \lambda_S = 1],\ \lambda_i < \lambda_{i+1}$), mapping of the original functions to an equivalent space leads to designing S sequences of scaled LSFs (with increasing safety domain) between these two LSFs (say $g(\boldsymbol{u};\lambda_1)$ and $g(\boldsymbol{u};1)$) [13,25]. Using this idea, the failure probability of the scaled LSFs can be written as follows:

$$P_{f(\Theta)} = P[g(\boldsymbol{u};\lambda_1) \le 0] > P[g(\boldsymbol{u};\lambda_2) \le 0] > \cdots > P[g(\boldsymbol{u};1) \le 0] = P_{f(\Omega)}, \tag{6}$$

This approach is shown in Fig. 3. The same implementations can be found in Ref. [13] for deriving the SORM formulation where it is assumed that the reliability index for $g(\boldsymbol{u};\lambda_1) = 0$ is $\beta = |\boldsymbol{u}^*| = 1$.

Instead of relying on the design points to solve the problem, Ref [25] suggested an approach for probability estimation using conditional probabilities and random sampling, focusing only on the sequence of intermediate functions. Considering $\mathbb{I}_{g(\boldsymbol{u};\lambda_1)}$ as the indicator function, the

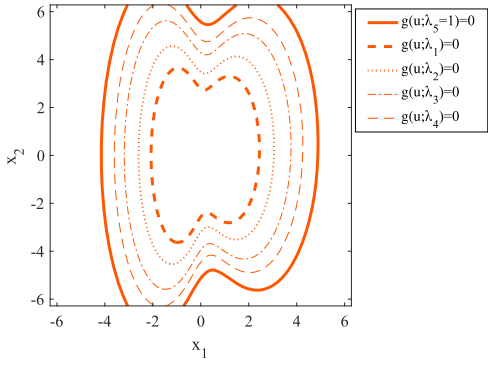

Fig. 3. Original ($g(\boldsymbol{u};\lambda_5 = 1)$) and scaled LSFs ($g(\boldsymbol{u};\lambda_i),\ i = 1:4$) of Metaball functions: The scale parameter λ increased from $\lambda_1 = 0.5$ to $\lambda_5 = 1$ which led to obtaining nested LSFs between the original LSF of $g(\boldsymbol{u}) = g(\boldsymbol{u};\lambda_5 = 1) = 0$ and the fictitious function of $g(\boldsymbol{u};\lambda_4 = 0.5) = 0$.

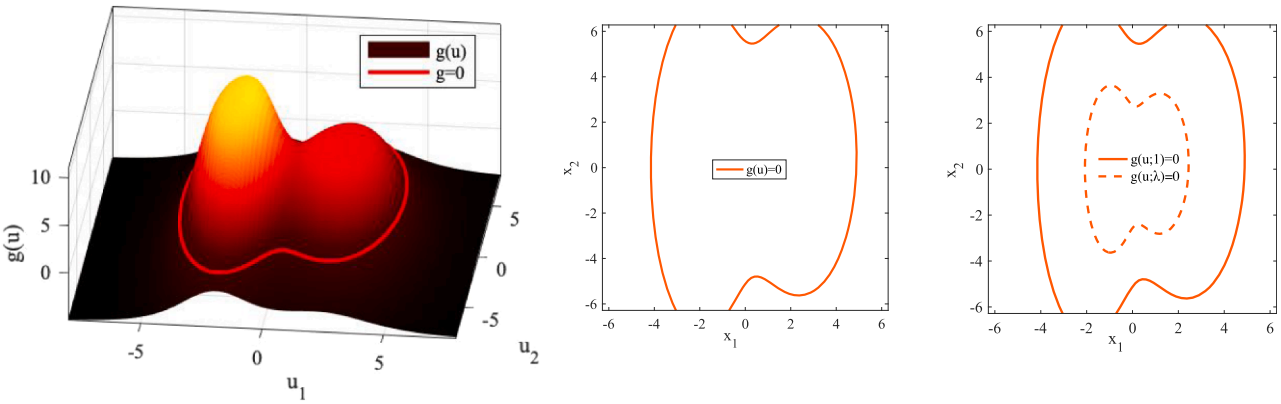

Fig. 2. The Metaball function, the original, and the scaled LSF $g(\boldsymbol{u};\lambda = 0.5) = g(0.5^{-1}\boldsymbol{u}) = 0$ with higher failure probability.

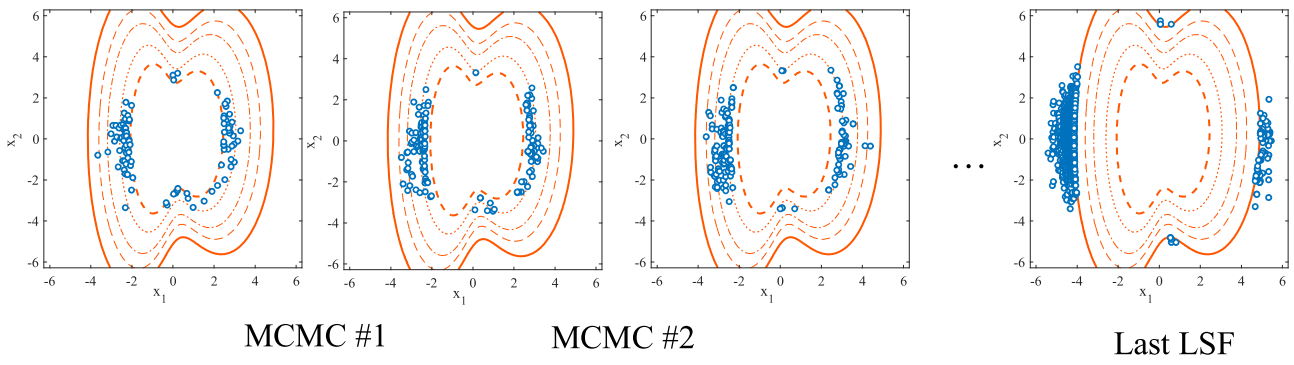

Fig. 4. Improved subset simulation (SESC): Employing scaled intermediate LSFs for the transition of MCMC samples toward failure regions.

first probability in Eq. (6) can be estimated using crude MCS as:

$$P_{f(\Theta)} = P(F_1) = P[g(\boldsymbol{u};\lambda_1) \le 0] = \int \mathbb{I}_{g(u;\lambda_1)}(\mathbf{u}) f(\boldsymbol{u}) \mathrm{d}u = \mathbb{E}\left(\mathbb{I}_{g(\boldsymbol{u};\lambda_1)}(\mathbf{u})\right). \quad (7)$$

Then, in the case of intermediate LSFs, one may estimate the conditional failure probabilities $P(F_i|F_{i-1})$ for intermediate failure domains $F_i = \{g(\boldsymbol{u};\lambda_i) \le 0\}$ using Markov Chain Monte Carlo (MCMC). This leads to a total approximation of L^{th} intermediate LSFs as follows:

$$P(F_L) = P(F_1) \prod_{i=2}^{L} P(F_i|F_{i-1}), \quad (8)$$

which represents the main formulation of the Sequential space conversion method (SESC) [25].

The traditional subset simulation (denoted as Sus) employs intermediate failure domains based on decreasing thresholds (g_i) using the geometry of performance function (i.e., $F_i = \{g(\boldsymbol{u};i) \le g_i\}$), and formulates the failure probability as:

$$P_f(\text{Sus}) = P(F_1) \prod_{i=2}^{S} P(F_i|F_{i-1}), F_i = \{g(\boldsymbol{u};i) \le g_i\}. \quad (9)$$

In contrast to this traditional treatment, as shown in Fig. 4, the improved subset simulation (denoted as ISus), uses scaled failure domains (i.e., $F_i = \{g(\boldsymbol{u};\lambda_i) \le 0\}$) and therefore its estimation of the total failure probability is independent of the geometry of performance function:

$$P_f(\text{ISus}) = P(F_1) \prod_{i=2}^{S} P(F_i|F_{i-1}),\ F_i = \{g(\boldsymbol{u};\lambda_i) \le 0\}, \quad (10)$$

which mathematically addresses the main concerns about the transition of MCMC toward important failure regions in the subset simulation method [25]. The different performances of traditional and improved subset simulation (i.e., using Eqs. (9) and (10), respectively) are investigated in Section 3.1, with details.

Remark 1. It is worth mentioning that the assumption behind the SESC (in Eq. (10)) is that the failure domain of scaled LSF is higher than the original failure domain and the safety domain of nested LSFs is increasing. However, scaling based on $\widehat{g}(\mathbf{u}) = g(\lambda^{-1}\boldsymbol{u})$ may not satisfy such a requirement for some problems with island failure domains. Therefore, a scaling approach fit to the specification of the in-hand problem (e.g., see Table 1) should be used in the analysis to form the nested LSFs.

In the proposed formulation, the failure probability of SESC can be presented as a function of the scale parameter λ, $P_f(\text{ISus}) = P_f(\lambda)$. In this case, the total approximation of failure probability is obtained when $\lambda = 1$.

This reformulation of subset simulation presents an analogy with the concept of asymptotic approximation in extreme value theory, which focuses on the asymptotic behaviors of multi-normal integrals in the tail of marginal functions [49,50,52,53].

Considering the point that in the proposed approach, estimation starts from λ_1 and terminates to $\lambda_S = 1$, it would be possible to shift analysis from a numerical to an approximation approach where extreme values theory provides an explicit approximation of the failure probability for $\lambda = 1$ as follows [49,50]:

$$P_f(\lambda) \approx \widetilde{q}(\lambda)\exp\{-a(\lambda - b)^c\}, \lambda \to 1, \quad (11)$$

where a, b, and c are constants and $\widetilde{q}(.)$ is a function that will behave very much like a constant [49]. In this approach, $P_f(\lambda)$ reflects the basic assumption of an asymptotic Gumbel distribution of the extremes which has proven enough flexible for all cases considered so far [49,50].

According to the proposed perspective, when the scale parameter approaches one, Eq. (10) and (11) present the same value (i.e., $P_f(\text{ISus}) \approx \widetilde{q}\cdot\exp\{-a(\lambda - b)^c\}$). Based on this analogy, one is only required to estimate a few terms of Eq. (6) and then, by estimating the parameters of Eq. (11), extrapolate the failure probability for λ=1. For this purpose, estimating the first T probabilities of Eq. (6), we may consider scale factors $\boldsymbol{\lambda} = [\lambda_1, \lambda_2, \ldots, \lambda_{T-1}, \lambda_T]$ and the corresponding failure probabilities $\boldsymbol{Y}$ (e.g., $Y(1) = P(F_1)$ and $Y(j) = P(F_1)\prod_{i=2}^{j} P(F_i|F_{i-1})$ where $j = 2, 3, \ldots, T$) as the input and output to estimate parameters of the Eq. (11) and extrapolate failure probability as follows:

$$P_f(\text{Asymptotic}) \approx q\exp\{-a(\lambda - b)^c\}, \quad (12)$$

where the parameters of the proposed extrapolation function, i.e., $\theta = (q, a, b, c)$, can be estimated by minimizing the mean square error between the log scale of two sides of Eq (12) as follows [50]:

$$\text{Find } \theta = (q,\ a, b,\ c) \quad (13)$$

$$\text{Minimize}: \sum_{i=1}^{T} \left(\text{Log}\left(P_f(\lambda_i)\right) - \text{Log}(q) + a(\lambda_i + b)^c\right)^2$$

Once the parameters are obtained through optimization, the total failure probability can be extrapolated for $\lambda = 1$ as follows:

$$P_f(\text{Asymptotic\#1}) \approx q\exp(-a(1 + b)^c), \quad (14)$$

The Asymptotic subset simulation (Asus) is schematically presented in Fig. 5.

Proposition 1. *The second-order reliability method (SORM) [13] is a specific version of the Eq. (11).*

Proof. Following the implementations presented in Section 2.1, for a problem with LSF of $g(\boldsymbol{u})$ and very small failure probability (i.e., $\beta_{g(\boldsymbol{u})} \to \infty$), one can present a weakened version of a system with the performance function of $\widehat{g}(u)$ and reliability index of $\beta_{\widehat{g}(\boldsymbol{u})} = 1$. Then, the

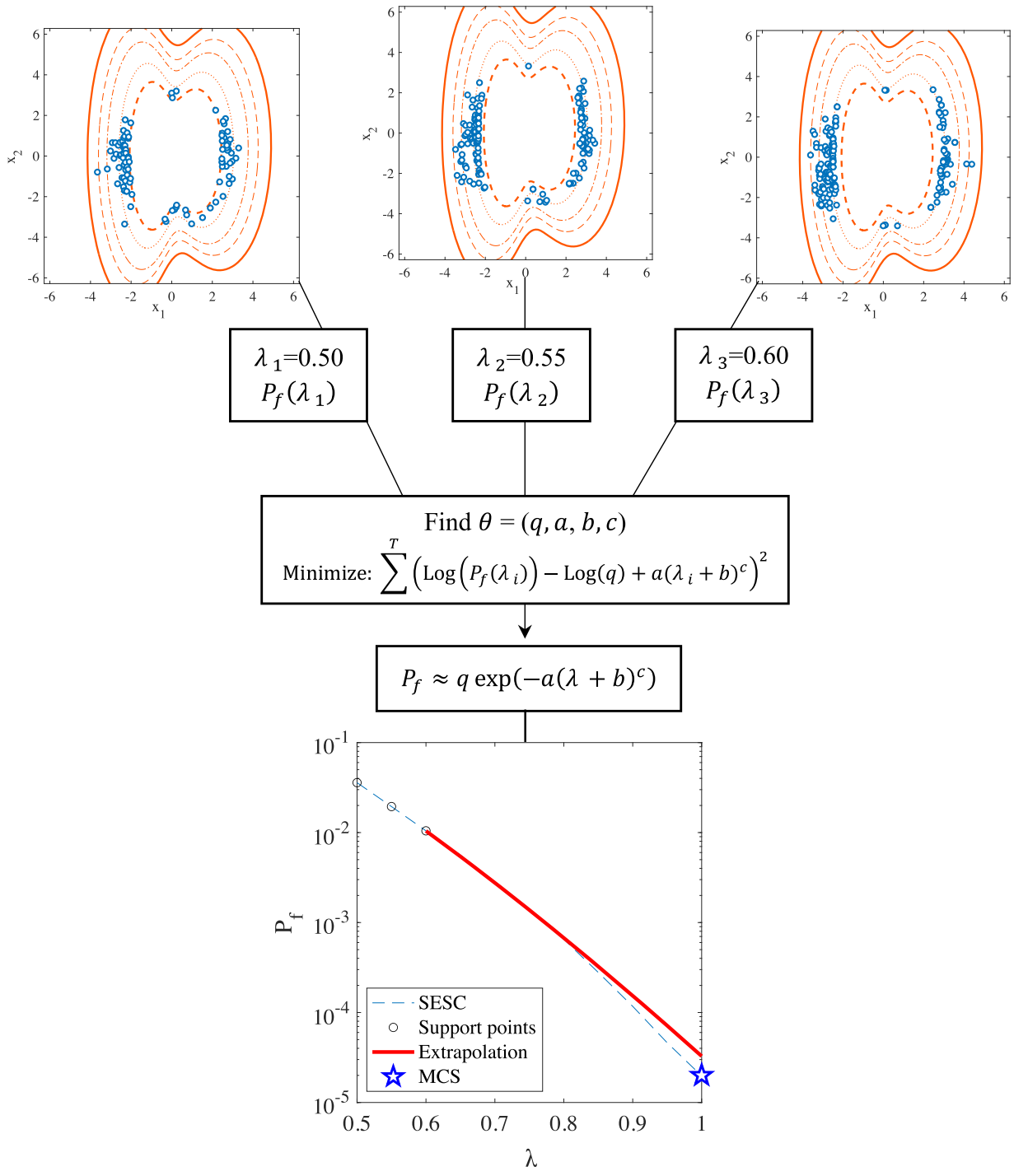

Fig. 5. Concept of Asymptotic subset simulation (Asus): By employing governing equations in extreme values theory and asymptotic approximation, only a few steps of subset simulation (with scaled LSFs) could be enough to extrapolate small failure probabilities.

scale parameter $\lambda = \frac{\beta_{g(u)}}{\beta_{g(u)}}$ can be used to design a set of nested LSFs with increasing safety domain (then, one obtains $\frac{1}{\beta_{g(u)}} < \lambda < 1$). On the other hand, for a random variable with standard normal PDF, using Mill's ratio, one gets:

$$\Phi(-\beta) \approx \frac{\beta}{\sqrt{2\pi}}.\exp\{-0.5\beta^2\}(\beta\to\infty). \tag{15}$$

As a result, by employing $a = 0.5$, $b = \beta + 1$, $c = 2$, and $q = \frac{\beta^{-1}}{\sqrt{2\pi}\cdot\sqrt{\prod_{i=1}^{n-1}(1-\kappa_i)}}$ (κ represents the curvature of function) in Eq. (11), one obtains:

$$P_f(\text{Asymptotic}) \approx \frac{\beta^{-1}}{\sqrt{2\pi}\cdot\sqrt{\prod_{i=1}^{n-1}(1-\kappa_i)}}\exp\{-0.5\beta^2\},\ \lambda\to 1. \tag{16}$$

that is the main formula of SORM:

$$P_f(\text{SORM}) \approx \frac{\beta^{-1}}{\sqrt{2\pi}\cdot\sqrt{\prod_{i=1}^{n-1}(1-\kappa_i)}}\exp\{-0.5\beta^2\} = \Phi(-\beta)|J|^{-1/2}, \tag{17}$$

where $J = \sqrt{\prod_{i=1}^{n-1}(1-\kappa_i)}$.

Using the proposed implementations, one may also review the concept of SORM formulation from the perspective of extreme values. As a main issue, by focusing on the scale factor $\lambda = \frac{\beta_{\hat{g}(u)}}{\beta_{g(u)}}$, one may find that the SORM in Eq. (17) is built upon the following (hidden) assumption: if $\beta_{g(u)} > \beta_{\hat{g}(u)}$ then $P(g(\boldsymbol{u}) \le 0) < P(\widehat{g}(\boldsymbol{u}) \le 0)$.

However, similar to that presented in Remark 1 for SESC, for the mentioned scaling approach, this assumption may no longer hold for some problems (e.g., island failure functions). For such functions, even if $\beta_{g(u)} > \beta_{\hat{g}(u)}$, we don't necessarily obtain $P(g(\boldsymbol{u}) \le 0) < P(\widehat{g}(\boldsymbol{u}) \le 0)$ which is the necessary condition of the asymptotic analysis.

Therefore, a richer formulation for asymptotic analysis requires the design of scaled LSFs based on the failure probabilities (and not the reliability index) using a scale approach that fits the specification of the problem (and not only one certain scaling approach). It is worth mentioning that, besides a suitable scaling approach, using the general reliability index in assumptions ($\beta_{g(u)} = -\Phi^{-1}(P_{f(\Omega)}) > -\Phi^{-1}(P_{f(\Theta)}) = \beta_{\hat{g}(u)}$) may also fix the issue.

Nonetheless, since SORM only uses the original LSF $g(u)$ in analysis (and not scaled LSFs), the mentioned formulation drawback does not affect its reliability results.

Remark 2. Proposition 1 and Section 2.1 reveal that the proposed asymptotic subset simulation and SORM share the same formulation and therefore, if we consider the random sampling in subset simulation (using a suitable scaling approach) as a kind of optimization approach looking for the important failure regions, a few steps of the (random) search process could be enough to approximate a small failure probability.

Proposition 2. *In case of problems where MCMC properly conducts random samples toward important failure regions, the extrapolation may be used in traditional subset simulation and sequential importance sampling to approximate the target failure probability*.

Proof. As presented in Eq. (9), traditional subset simulation employs intermediate failure domains based on the geometry of the performance function $F_i = \{g(\boldsymbol{u}; i) \le g_i\}$. In the first step of this method, let's consider $F_1 = \{g(\boldsymbol{u}; 1) \le g_1\}$ with threshold g_1 as the failure domain Θ in Eq. (3):

$$P_{f(\Theta)} = P(F_1) = P[g(\boldsymbol{u}; 1) \le g_1] = \int_{\Theta=\{g(\boldsymbol{u};1)\le g_1\}} f(\boldsymbol{u})d\boldsymbol{u} = \mathbb{E}\left(\mathbb{I}_{g(\boldsymbol{u};1)}(\boldsymbol{u})\right). \tag{18}$$

Then, by using intermediate thresholds g_i $\{i = 1, 2, \ldots, m\}$, one may define the scale parameter λ_i as a function of g_i as follows:

$$\lambda_i = \left(\frac{1}{\frac{g_i}{g_1}+1}\right)^{0.25}. \tag{19}$$

In the proposed equation, λ_i starts from $\lambda_1 = 0.84$ (since in the first step, we obtain $\lambda_1 = \left(\frac{1}{1+1}\right)^{0.25} = 0.84$) and after m steps, when the threshold g_i meets the original failure domain, λ_i approaches to one (i.e., when $g_m = 0$, we obtain $\lambda_i = \left(\frac{1}{0+1}\right)^{0.25} = 1$). Therefore, just by performing a few steps of traditional subset simulation, one may form two vectors for λ and $P_f(\lambda)$ to use the extrapolation formula of Eq. (11) for $\lambda_i = 1$ (which corresponds to $g_m = 0$):

$$P_f(\text{Sus}) = P[g(\boldsymbol{u}; m) \le 0] \approx \widetilde{q}\cdot\exp\{-a(1-b)^c\}, \lambda \to 1. \tag{20}$$

The same implementation can be presented for sequential importance sampling methods. To meet this aim, let's call the $P_{f(\Theta)}$ of Eq. (3) the "relaxed failure probability" which is easy to estimate, and call the scale factor parameter λ the "relaxation parameter". Then, as explained in Ref. [54] in detail, by presenting $\eta_j(\boldsymbol{u}; \lambda_j) = \mathbb{I}_{g(\boldsymbol{u})\le 0}(\boldsymbol{u})f(\boldsymbol{u}; \lambda_j)$, $j = 1, 2, \ldots, S$ and employing an optimal probability density function as follows:

$$k_j^*(\boldsymbol{u}; \lambda_j) = \frac{\eta_j(\boldsymbol{u}; \lambda_j)}{P_j} = \frac{\mathbb{I}_{g(\boldsymbol{u})\le 0}(\boldsymbol{u})f(\boldsymbol{u}; \lambda_j)}{\int \mathbb{I}_{g(\boldsymbol{u})\le 0}(\boldsymbol{u})f(\boldsymbol{u}; \lambda_j)d\boldsymbol{u}}, j = 1, 2, \ldots, T. \tag{21}$$

Here, one may design a set of "less relaxed" (say intermediate) probabilities between the $P_{f(\Theta)}$ and $P_{f(\Omega)}$ (for $\boldsymbol{\lambda} = [\lambda_1, \lambda_2, \ldots, \lambda_{S-1}, \lambda_S = 1]$) and use sequential sampling to provide an approximation of failure probability as a function of the parameter λ:

$$P_f(\text{IS}) = P_f(\lambda) = P(F_1)\prod_{i=1}^{S-1}\frac{f(\boldsymbol{u}; \lambda_{j+1})}{f(\boldsymbol{u}; \lambda_j)}k_j^*(\boldsymbol{u}; \lambda_j). \tag{22}$$

As a result, according to the explanations presented for subset simulation, by estimating a few terms of Eq. (14), one may use the first T parameters λ (e.g., $\lambda_1 > \lambda_2 > \ldots > \lambda_T$) and corresponding probabilities (e.g., $P(F_1) > P(F_2) > \ldots > P(F_T) = P(F_1)\prod_{i=1}^{T-1}\frac{f(\boldsymbol{u}; \lambda_{j+1})}{f(\boldsymbol{u}; \lambda_j)}k_j^*(\boldsymbol{u}; \lambda_j)$) as input and use Eq. (11) to extrapolate the total failure probability for $\lambda_S = 1$.

The same implementations may be used for different versions of sequential sampling methods [36].

We should note that using a kernel density function, an attempt to extrapolate the PDF of marginal function by traditional subset simulation is proposed in Ref [55] (which is technically and conceptually different from the suggested approach). However, the main problem regarding using traditional subset simulation (and sequential importance sampling) for approximation of the total failure probability is that the topology may misconduct MCMC samples to find important failure domains (as clarified in Ref. [37]). Besides, in the case of Proposition 2, the scaling depends on the topology of the performance function which may be very different in the vicinity of the origin and the LSF.

Therefore, although extrapolation of traditional subset simulation may properly work for many problems, it fails to solve intricate functions and the formulations explained in Proposition 2 cannot be considered as a general solution for Eqs. (1) and (2).

2.2. Alternative extrapolation functions and ensemble averaging model

In the proposed framework, Eq. (12) represents a four-parameter Gumbel distribution for the approximation of the tail of the marginal function in the failure domain and, as explained in Ref. [49], the mentioned equation is flexible for all (rare event) cases considered so far. Nonetheless, there are some other alternative functions (with fewer/more parameters) that may be used for extrapolation. For instance, the following function is suggested by Butcher [45] to exploit the asymptotic behavior of the generalized reliability index of scaled LSFs:

$$\beta(\lambda) \approx a\lambda + \frac{b}{\lambda^c}. \tag{23}$$

The parameters $\theta = (a, b, c)$ in Eq. (23) can be obtained by using a parameter estimation approach for a few scale factors and corresponding probabilities obtained from Eq. (10). Then, asymptotic approximation for the problem can be estimated as:

$$P_f(\text{Asymptotic\#2}) \approx \Phi(-a-b). \tag{24}$$

Separate from this approach, the comparison of the SORM formulation and the extrapolation function of extreme values provides the opportunity to study Eq. (11) using the results of Proposition 2. For example, the comparison reveals that $\widetilde{q}(\lambda)$ in Eq. (11) mainly applies the effect of the curvature of the LSF function (at the important failure region) to the total probability while the exponential term $\exp\{-$

Table 2
Suggested extrapolation functions for failure probability approximation.

	Extrapolation Function $P_{f(\lambda)}$	parameters	Cost function for parameter estimation using λ_i and $P_f(\lambda_i)$
1	$q\exp(-a(\lambda_i+b)^c)$ [49]	$(q,\ a,\ b,\ c)$	$\sum_{i=1}^{T}\left(\mathrm{Log}(P_f(\lambda_i))-\mathrm{Log}(q)+a(\lambda_i+b)^c\right)^2$
2	$\Phi(-a-b)$ [45,56]	$(a,\ b)$	$\beta(\lambda)\approx a\lambda+\frac{b}{\lambda^c}$
3	$\frac{\exp(-a(\lambda+b)^c)}{\sqrt{2\pi}\cdot(\lambda+b)\cdot\not{\jmath}^e}$	$(a,\ b,\ c,\not{\jmath},e)$	$\sum_{i=1}^{T}\left(\mathrm{Log}(P_f(\lambda_i))-\mathrm{Log}\left(\frac{(\lambda_i+b)^{-1}}{\sqrt{2\pi}\not{\jmath}^e}\right)+a(\lambda_i+b)^c\right)^2$
4	$\frac{\exp\left(-\frac{\lvert\lambda_i\cdot b\rvert^2}{2}\right)}{\sqrt{2\pi}\cdot a\cdot\lambda_i\cdot b}$	$(a,\ b)$	$\sum_{i=1}^{T}\left(\mathrm{Log}(P_f(\lambda_i))-\mathrm{Log}\left(\frac{(\lambda_i\cdot b)^{-1}}{\sqrt{2\pi}a}\right)+\frac{\lvert\lambda_i\cdot b\rvert^2}{2}\right)^2$
5	$\frac{\exp\left(-\frac{\lvert\lambda_i\cdot b\rvert^2}{2}\right)}{\sqrt{2\pi\lambda_i\cdot b\cdot a}}$	$(a,\ b)$	$\sum_{i=1}^{T}\left(\mathrm{Log}(P_f(\lambda))-\mathrm{Log}\left((2\pi\lambda_i\cdot b\cdot a)^{-\frac{1}{2}}\right)+\frac{\lvert\lambda_i\cdot b\rvert^2}{2}\right)^2$
6	$\frac{\exp\left(-\lvert\lambda_i\cdot b\rvert^2\right)}{\sqrt{2\pi\cdot b\cdot a}}$	$(a,\ b)$	$\sum_{i=1}^{T}\left(\mathrm{Log}(P_f(\lambda_i))-\mathrm{Log}\left((2\pi\cdot b\cdot a)^{-\frac{1}{2}}\right)+\lvert\lambda_i\cdot b\rvert^2\right)^2$
7	$\exp(q)\exp(-a(\lambda_i+b)^c)$	$(q,\ a,\ b,\ c)$	$\sum_{i=1}^{T}\left(\mathrm{Log}(P_f(\lambda_i))-\mathrm{q}+a(\lambda_i+b)^c\right)^2$

$a(\lambda-b)^c\}$ mainly applies the effect of distance of the important failure to the origin. Therefore, by employing the results of Proposition 1, one may conclude that $\widetilde{q}(\lambda)$ is also a function proportional to the inverse of the reliability index (the reliability index is also related to the parameter b in Eq. (11)) and therefore, it would be more appropriate to represent the general form of Eq. (11) as follows:

$$P_f(\lambda)\approx\widetilde{q}(\lambda,b)\exp\{-a(\lambda-b)^c\},\lambda\to1 \tag{25}$$

On the other hand, since the curvature of LSFs is a function of the scale parameter λ, this study also investigates the following function for extrapolation:

$$P_f(\lambda)\approx\widetilde{q}(\lambda,b)\cdot\exp(-a(\lambda+b)^c),\ \lambda\to1 \tag{26}$$

$$\widetilde{q}(\lambda,b)=\frac{(\lambda+b)^{-1}}{\sqrt{2\pi}\not{\jmath}^e},$$

where $\theta=(a,\ b,\ c,\not{\jmath},\ e)$ are the parameters of the proposed extrapolation function which can be estimated using the following optimization process:

$$\text{Find }\theta=(a,b,c,\not{\jmath},\ e) \tag{27}$$

$$\text{Minimize}:\ \sum_{i=1}^{T}\left(\mathrm{Log}(P_f(\lambda_i))-\mathrm{Log}\left(\frac{(\lambda_i+b)^{-1}}{\sqrt{2\pi}\not{\jmath}^e}\right)+a(\lambda_i+b)^c\right)^2$$

Then, the failure probability can be extrapolated for $\lambda=1$ as follows:

$$P_f(\text{Asymptotic\#3})\approx\frac{(1+b)^{-1}}{\sqrt{2\pi}\not{\jmath}^e}\cdot\exp(-a(1+b)^c). \tag{28}$$

Using such functions provides the opportunity to obtain more information about the problem at hand. For instance, we also use fixed values for parameters a and c as 0.5 and 2, and employed the following function for extrapolation where the scale parameter λ_i is multiplied to the parameter b as follows:

$$P_f(\lambda)\approx\frac{b^{-1}}{\sqrt{2\pi}(\lambda_i\cdot a)}\cdot\exp\left(-\frac{\lvert\lambda_i\cdot b\rvert^2}{2}\right),\ \lambda\to1 \tag{29}$$

that results in the following asymptotic approximation of the failure probability:

$$P_f(\text{Asymptotic\#4})\approx\frac{b^{-1}}{\sqrt{2\pi}\cdot a}\cdot\exp\left(-\frac{\lvert b\rvert^2}{2}\right), \tag{30}$$

which comparison of Eq. (30) with SORM may be used to employ extrapolation to exploit some information about the parameters of the problem. For example, for problems with one design point, we get $|\boldsymbol{u}^*|=b$ and $\sqrt{\prod_{i=1}^{n-1}(1-\kappa_i)}=\frac{P_f(\text{Asymptotic\#4})}{\Phi(-b)}$ that represents the reliability index and curvature of the LSF at the design point, respectively.

Besides the mentioned extrapolation functions, three other asymptotic formulas are also suggested in this study (See Table 2) and their parameters can be estimated according to the implementations proposed in this section. Here, because each extrapolation function may tend to underestimate/overestimate the true failure probability of the in-hand problem, the average ensemble model of the proposed functions is suggested as the total approximation of the failure probability by the proposed approach:

$$P_f\approx\mathbb{E}\left(P_{f(\lambda)}\right) \tag{31}$$

In cases where a certain extrapolation function presents results too far from other functions, the practitioner may delete the result of the mentioned function from the analysis.

2.3. Generalization of the framework

A failure probability is generally a function of parameters $\boldsymbol{\theta}=(\theta_1,\ldots,\theta_k)$ and, as explained in [57] detail, a richer presentation of the failure probability would be as follows:

$$P(\mathrm{F}|\theta)=\int_{\{g(\boldsymbol{u},\theta)\le0\}}f(\boldsymbol{u}|\boldsymbol{\theta})\mathrm{d}\boldsymbol{u}. \tag{32}$$

Let's assume that we have changed the parameters of the problem from $\boldsymbol{\theta}$ to $\boldsymbol{\theta}'$ for obtaining a weakened version of the system with a higher failure probability. This change would lead to obtaining a new failure probability as:

$$P(\mathrm{F}|\theta')=\int_{\{g(\boldsymbol{u},\boldsymbol{\theta}')\le0\}}f(\boldsymbol{u}|\boldsymbol{\theta}')\mathrm{d}\boldsymbol{u}. \tag{33}$$

In this formulation, as explained in Section 2.1, it is aimed to have two main specifications:

1) By solving Eq. (33), we obtain $P(\mathrm{F}|\boldsymbol{\theta}')\gg P(\mathrm{F}|\boldsymbol{\theta})$.
2) The original failure domain (in standard normal space) is a subset of the new failure domain $g(\boldsymbol{u},\boldsymbol{\theta}')\le0$, and the new LSF reflects a scaled version of the original LSF.

Based on the specification of the in-hand problem, different scaling

approaches can be used for obtaining $P(\mathrm{F}|\theta')$ with LSF of $g(\boldsymbol{u},\boldsymbol{\theta}')$. Also for this case, a few approaches are suggested in Table 1. Using the mentioned implementations, compared to the $P(\mathrm{F}|\theta)$, the failure probability $P(\mathrm{F}|\theta')$ can be very efficiently approximated by crude MCS.

By mapping this problem into standard normal space, since $\frac{1}{\sqrt[n]{2\pi}}\exp\left\{-\frac{1}{2}|\boldsymbol{u}|^2\right\}$ is not a function of $\boldsymbol{\theta}'$, the dependence of the failure probability on the parameter $\boldsymbol{\theta}'$ appears only in the LSF and therefore, this generates a new LSF for the problem which can be presented as $\widehat{g}(\mathbf{u},\boldsymbol{\theta}')$ (one also needs to map the failed samples obtained in Eq. (33) to the new space). Now, it is only required to present parameter vector $\boldsymbol{\theta}'$ as a function of λ, denoted as $\boldsymbol{\theta}'(\lambda)$, in such a manner that for $\lambda = 1$ we get $\boldsymbol{\theta}'(1) = \boldsymbol{\theta}$ (i.e., if $\lambda \to 1$, then $\boldsymbol{\theta}' \to \boldsymbol{\theta}$). For such a situation, the corresponding failure probability for any parameter λ_L can be estimated using SESC formulation:

$$P(\mathrm{F}|\theta'(\lambda_{(L)})) = P(\mathrm{F}|\theta'(\lambda_{(1)}))\prod_{i=2}^{L} P(F_i|F_{i-1}),\ F_i = \{g(u,\boldsymbol{\theta}'(\lambda_{(t)})) \le 0\}. \tag{34}$$

Therefore, for each value in the vector of $\boldsymbol{\lambda} = [\lambda_1,\lambda_2,\ldots,\lambda_T]$, we obtain a corresponding failure probability as $\boldsymbol{P}(\mathrm{F}|\theta'(\lambda)) = [P(\mathrm{F}|\theta'(\lambda_{(1)})), P(\mathrm{F}|\theta'(\lambda_{(2)})),\ldots, P(\mathrm{F}|\theta'(\lambda_{(T)}))]$. Then, according to the implementations presented in the former section, the vector $\boldsymbol{\lambda}$ and its corresponding failure probabilities can be used to extrapolate the small probability $P(\mathrm{F}|\boldsymbol{\theta}'(1)) = P(\mathrm{F}|\theta)$ for $\lambda = 1$ using the following general formula (See Section 2.2, for more suggestions):

$$P(\mathrm{F}|\theta'(\lambda)) \approx \widetilde{q}(\lambda)\exp\{-a(\lambda-b)^c\}, \lambda \to 1, \tag{35}$$

where this study suggests using the ensemble of alternative extrapolation functions of Table 2 to approximate the total failure probability of the problem, noting the point that when an extrapolation model fails to properly determine the probabilities of the employed support points, it is required to delete the result of the mentioned function from the analysis.

2.4. Statistical properties of estimation

Reliability analysis by the proposed approach involves performing two main processes, namely, (1) subset simulation using scaled LSFs (SESC approach) and, (2) considering failure probabilities of a few scaled SLFs as the support points to extrapolate total failure probability by the ensemble of asymptotic formulas provided in Table 2. To estimate the confidence intervals of the proposed approach, one may estimate the confidence intervals for each extrapolation function, separately, and then use the minimum of all obtained interval bounds as the lower bound of total probability (denoted by CI^-) and the maximum values of the confidence interval as the upper bound as CI^+. In this context, the confidence interval of each extrapolation function can be estimated as follows:

If one conducts the crude Monte Carlo for estimation of $P(F_1)$, the coefficient of variation δ_1 can be estimated as follows:

$$\delta_1 = \sqrt{\frac{1-P(F_1)}{NP(F_1)}}, \tag{36}$$

where N represents the total number of samples in the step. As presented in [58], in the case of sequential sampling, the δ_j of conditional probabilities $P(F_j|F_{j-1})\ j = 2,\ldots,\ M$ can be estimated as:

$$\delta_j = \sqrt{\frac{1-P(F_j)}{NP(F_j)}\left(1+\gamma_j\right)}, \tag{37}$$

where

$$\gamma_k = 2\sum_{k=1}^{\frac{N}{N_S}-1}\left(1-\frac{kN_S}{N}\right)\rho_j(k). \tag{38}$$

where N_S is the number of seeds of MCMC and $\rho_j(k)$ is the average k-lag auto-correlation coefficient of the stationary sequences and can be estimated from the samples [58]. By assumption of independence of the conditional probabilities, the coefficient of variation of failure probability, Cv, of both SESC and traditional subset simulation can be estimated as [24]:

$$\mathrm{Cv}^2 = \sum_{j=1}^{M}\delta_j^2. \tag{39}$$

Noting the point that the proposed estimation tends to underestimate the true coefficient of variation of failure probability [58,24]. Also in this study, one gets the following approximation for estimation of Cv for the failure probabilities obtained by simulation (for use as the support points for extrapolation):

$$\mathrm{Cv}^2 = \frac{1-P(F_1)}{NP(F_1)} + \sum_{j=2}^{M}\delta_j^2. \tag{40}$$

For estimation of the confidence interval for a predicted value of the failure probability provided by the optimal curve, once the $\mathrm{Cv}(\lambda)$ is estimated for T support points, one may use $\boldsymbol{\lambda} = [\lambda_2,\ldots,\lambda_{T-1},\lambda_T]$ as input and the corresponding $\mathrm{Cv}(\lambda)$ as output (i.e., $Y(j) = \sum_{i=1}^{j}\delta_i^2$ where j =2: T) to extrapolate $\mathrm{Cv}(\lambda)$ for $\lambda = 1$ using linear regression (the reason is that the parameters in Eqs. (37) and (38) are almost fixed values in each subset and then, one may find $\mathrm{Cv}(\lambda)$ follows the linear function for j=2 to m conditional probabilities). Assuming that sample data is normally distributed, a fair approximation of a 95 % confidence interval of each predicted probability can be estimated as $CI_{95\%}(\lambda) = [C^-(\lambda), C^+(\lambda)]$:

$$C^{\pm}(\lambda) = \widehat{P}_f(\lambda)\left(1 \pm 1.96\cdot\mathrm{Cv}\left(\widehat{P}_f(\lambda)\right)\right). \tag{41}$$

Noting the fact that in engineering applications, the upper bound of probability (minimum Reliability index) is more important than the lower bound, one may neglect the potential error associated with very small probabilities (See Ref. [57] for more detail) by restricting the probability in the lower bound to zero.

As explained former, once the confidence interval for each extrapolation function is estimated by the abovementioned approach, one may use the minimum and maximum of the obtained values as the confidence interval bounds of total failure probability.

3. Verification from the No-Free-Lunch perspective

According to the No-Free-Lunch theory in reliability analysis (NFLR) [38], no universal algorithm performs well across all problem types and, if a method is effective for a specific class of problems, it will perform poorly on the remaining problems. In this section, we analyze both the strengths and limitations of the proposed approach in various reliability problems. We present cases where it performs well, identify examples where state-of-the-art methods may outperform it, and provide a conceptual comparison with existing approaches.

In the optimization process of Eq. (13), the lower bound and upper bounds of parameters $\theta = (q,\ a,\ b,\ c)$ are considered as LB=[0,0,-8,0]; UB=[20,20,0,4], respectively. Furthermore, in the case of Eq. (26), LB=[0, 0] and UB=[20, 5] are used for the parameters of $\mathscr{f}$ and e.

3.1. Evaluating the efficiency of the proposed framework

The approach #1 is used for scaling the performance function of this subsection (see Table 1) where the LSF is scaled as $g(\lambda\boldsymbol{u})$. The initial scale parameter is set to $\lambda_1 = 0.5$ and the other parameters including the

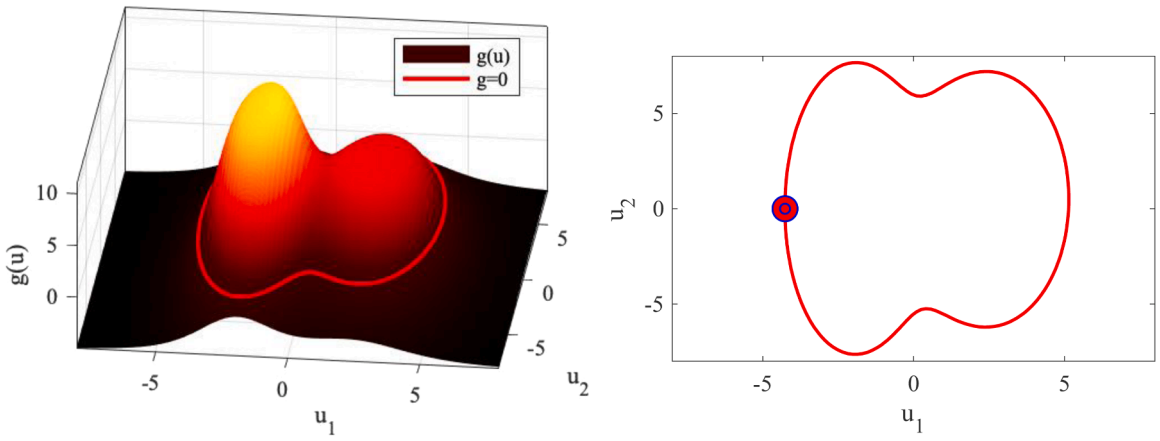

Fig. 6. Left: The geometry of the Metaball function (with parameter a=1) and Right: the LSF and most important failure region of the problem.

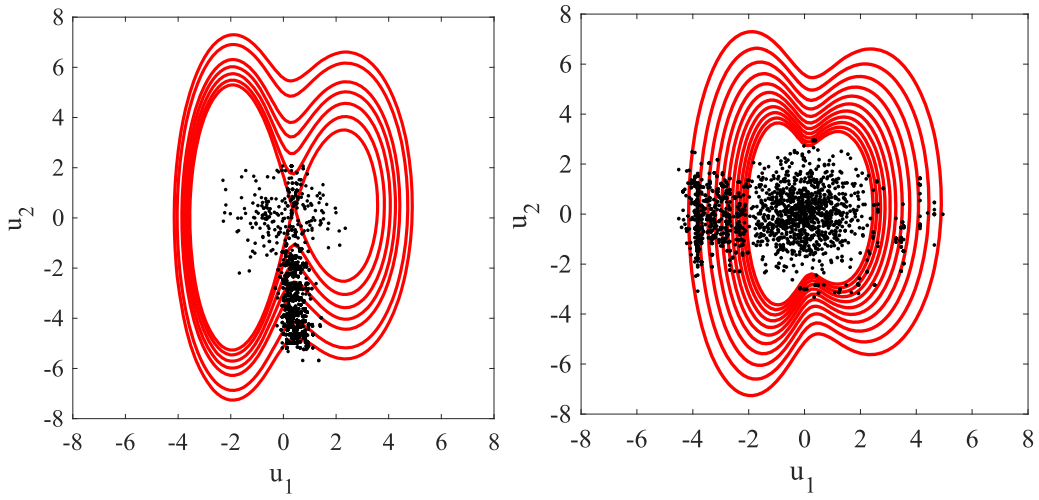

Fig. 7. Left: Traditional subset simulation and sequential IS fail to properly solve the Metaball problem. Right: Improved subset simulation (SESC) with scaled LSFs as intermediate thresholds.

number of support points are reported for each example, separately. As explained in Section 2.2, the ensemble average of the suggested extrapolation function is used as the total approximation of the failure probability.

3.1.1. *Intricate numerical example*

In the first example, we aim to show the efficiency of the proposed approach compared to traditional/improved subset simulation and asymptotic sampling. To meet this aim, reliability analysis of the Metaball test problem [37] with the standard normal random variables and the following LSF is investigated:

$$g(u_1,u_2)=\frac{30}{\left(\frac{4(u_1+2)^2}{9}+\frac{u_2^2}{25}\right)^2+\mathrm{a}}+\frac{20}{\left(\frac{(u_1-2.5)^2}{4}+\frac{(u_2-0.5)^2}{25}\right)^2+\mathrm{a}}-5, \qquad (42)$$

The problem is studied for two situations, namely: Case 1, for $\mathrm{a}=1$, and, Case 2, for $\mathrm{a}=2$ (a is the parameter of the denominator in Eq. (42)). The main specification of this example is having several important failure regions in different directions and also, its intricate geometry of the performance function that produces difficulties for search algorithms to find most important failure region of the problem (See Fig. 6).

This problem is considered in Ref [37] as a counter-example of the subset simulation since, as shown in Fig. 7, the traditional method (and also sequential IS [36]) fails to provide a proper approximation of the failure probability (for both Cases 1 and 2). The reason is that the topology of the performance function conducts MCMC samples toward the un-important failure region of the problem which results in an improper approximation of total failure probability.

However, as clarified in [25] and shown in Fig. 7, performing subset simulation based on the SESC approach addresses the above-mentioned drawback of the traditional approach, and using scaled LSFs would lead to transitions of MCMC toward the important failure region of the problem. As previously illustrated in Fig. 5 and reported in Table 3, by employing only a few steps of improved subset simulation, the proposed approach provides a suitable approximation of the failure probability with a portion of the function call required in the SESC approach. Besides, the function call of the method would be appropriately less than the asymptotic sampling [45] (less than 1/3) which requires performing

Table 3
The reliability results of the Metaball function for parameter a=2 (the same accuracy and function call obtained for a=1).

Method	P_f	β	#g-call
MCS	1.5×10^{-5}	4.17	10^7
Subset simulation	2.58×10^{-7}	5.02	5500
Sequential IS	1.69×10^{-7}	5.01	10,000
SESC	1.55×10^{-5}	4.16	7321
Asymptotic sampling	8.28×10^{-6}	4.31	9000
Asus	1.21×10^{-5}	4.22	2367

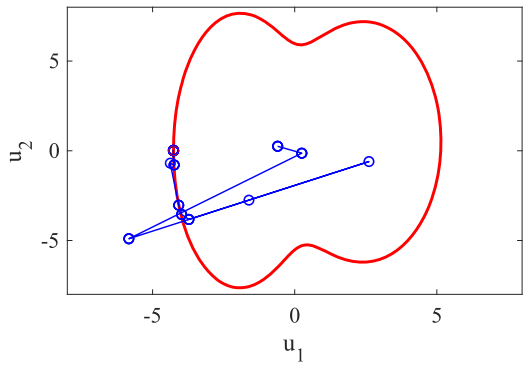

Fig. 8. Performing a gradient-based optimization algorithm from the origin would lead to properly finding important failure regions of Case 1 of the Metaball function (Eq. 42).

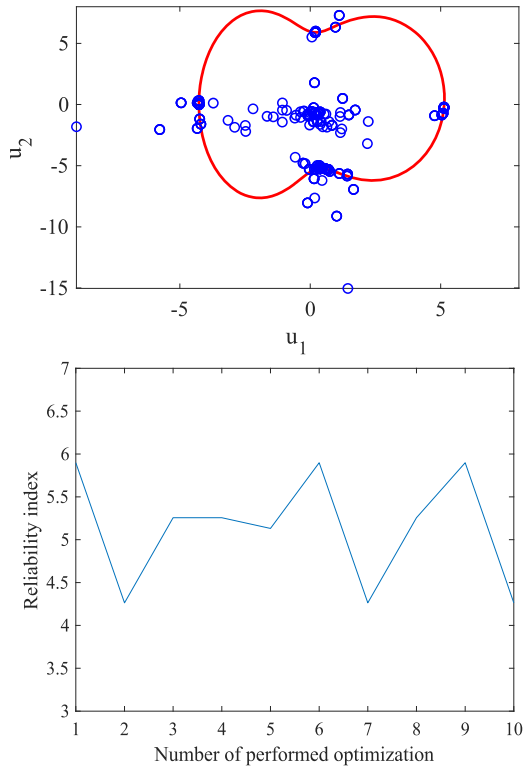

Fig. 9. History of 10 different optimization processes with random initial search points for Case 1 of the Metaball function.

separate MCS (for scale values of 0.25, 0.33, 0.4, and, 0.5) to obtain support points.

In Table 3, we have not reported the results of design point-based methods mainly because the accuracy of the obtained results and the number of function calls of these approaches highly depend on the decisions/skills of the practitioner. To clarify this issue, in the following, we investigated the Metaball problem for two mentioned cases:

- Case 1 with a=1 in Eq. (42), as follows:

$$g(u_1,u_2)=\frac{30}{\left(\frac{4(u_1+2)^2}{9}+\frac{u_2^2}{25}\right)^2+1}+\frac{20}{\left(\frac{(u_1-2.5)^2}{4}+\frac{(u_2-0.5)^2}{25}\right)^2+1}-5, \tag{42A}$$

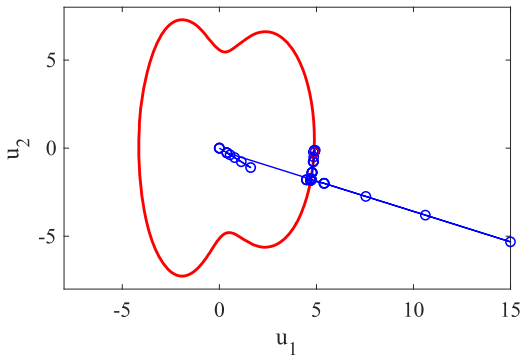

Fig. 10. History of design point search process for Case 2, initialized from the origin, which results in a wrong approximation of the important failure region.

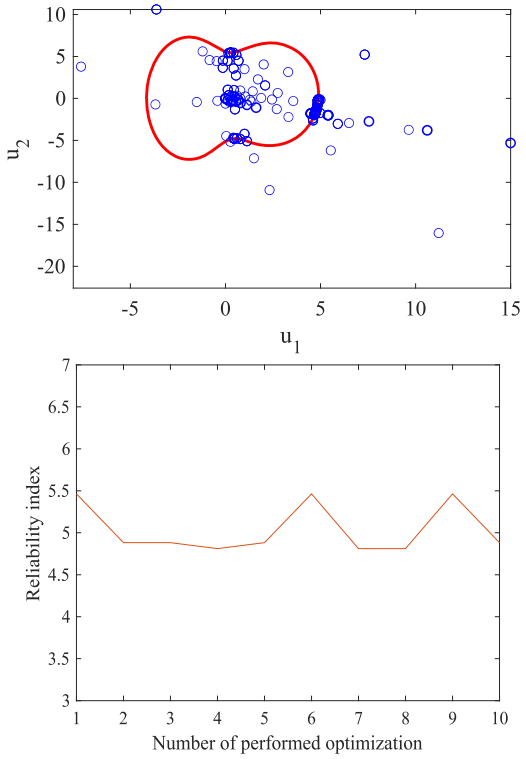

Fig. 11. History of 10 times optimization process with random initial search points for Case 2 of Metaball function: a wrong approximation of important failure region and failure probability.

For solving the proposed example using a design point-based approach, a practitioner (say P#1) may decide to start the search process from the origin for a gradient-based optimization approach (e.g., SQP method) which is a popular approach in reliability analysis). As shown in Fig. 8, this idea would lead to properly finding the important failure region of the problem. Then, Line sampling, importance sampling, or SORM may be used in analysis to apply the effect of the curvature in analysis (e.g., by addition of around 500~1000 function calls in estimation) which would lead to a suitable approximation of the failure probability of the problem.

In this case, if a practitioner (say P#2) initializes the optimization process from different random points (that is a wise approach in reliability analysis), obtains four different design points in different directions within about 600 function calls (See Fig. 9). Then, since one is not aware of the shape of LSFs around each design point and it is not clear which domains will have maximum participation in the total failure probability, it would be required to perform three extra

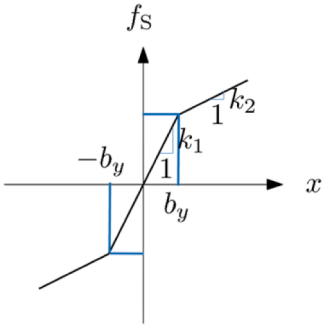

Fig. 12. Bilinear restoring force in the SDOF problem.

postprocessing (e.g., Line sampling, importance sampling or SORM) for the other design points to check the curvature and obtain probability. This would lead to an approximation of the failure probability with much more functional calls compared to the decision of P#1.

- Case 2, with a=2 in the denominator, as follows:

$$g(u_1,u_2)=\frac{30}{\left(\frac{4(u_1+2)^2}{9}+\frac{u_2^2}{25}\right)^2+2}+\frac{20}{\left(\frac{(u_1-2.5)^2}{4}+\frac{(u_2-0.5)^2}{25}\right)^2+2}-5, \tag{42B}$$

In this case, in contrast with Case 1, not only starting a gradient-based search process from the origin but also the addition of 10 times optimization from different random points would not be enough to find important failure region of the problem for probability estimation (See Figs. 10 and 11). Therefore, in Case 2, relaying on the former experience of Metaball functions (i.e., Case 1) would lead to a wrong approximation of failure probability and proper reliability analysis requires optimization with random initial points which considerably increases the function call of analysis.

This example clearly shows that, in case of such problems with intricate geometry and nonlinear LSFs, not only one search process but also 10 times optimization may not be enough/wise in reliability analysis. On the other hand, there is no clear answer to this question: how many optimization processes with random initial points are required to find all design points of a problem (which highlights the requirement of having a skilled optimization practitioner in reliability analysis)? Besides, having several design points in hand, performing postprocessing by SORM/Line sampling would be necessary for probability approximation which considerably increases the function call of analysis.

Using the proposed implementations, we also aim to clarify major drawbacks of employing gradient-based methods in analysis: 1) even by performing a huge number of function calls, there is no guarantee of finding all important failure regions of a problem and therefore, the accuracy of a design point-based method may be always in doubt, 2) the (huge) number of function calls required in search process cannot help the practitioner to obtain information about the curvature of LSF. For instance, while many points are required in the design point search process to discover failure regions, the function calls and information obtained in the search process are useless in SORM/Line sampling and the method disregards these function calls in their probability estimations.

However, if we consider the random sampling process in subset simulation as a kind of search process (e.g., see [38,53]), as explained in Section 2 and shown by this example, the suggested asymptotic approach uses the information obtained during a few steps of the search process to approximate small failure probabilities.

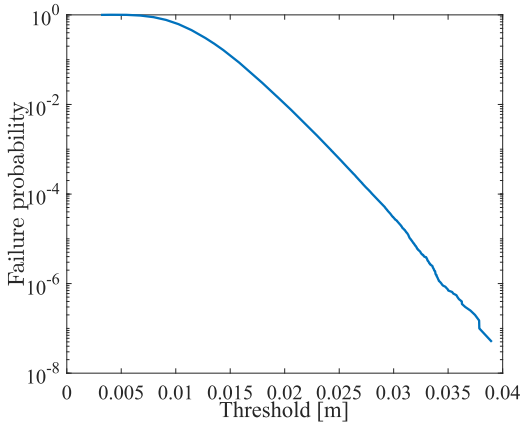

Fig. 13. Results of MCS for SDOF non-linear oscillator.

Table 4
The reliability results of the SDOF non-linear oscillator.

Method	P_f	β	#g-call
MCS	6.95×10^{-6}	4.35	2×10^7
Subset simulation	8.19×10^{-6}	4.31	8693
Sequential IS	6.14×10^{-6}	4.37	6000
SESC	7.87×10^{-6}	4.45	5456
Asus (15 NSPs*)	3.91×10^{-6}	4.47	2523
Asus (8 NSPs)	1.17×10^{-6}	4.57	1319

* NSPs=Number of support points

3.1.2. *High-dimensional engineering problem*

This example investigates the performance of the proposed framework for a problem with 1501 dimensions: calculating the first excursion probability of a single-degree-of-freedom (SDOF) non-linear oscillator subject to a stochastic force. The SDOF is assumed to possess a bilinear conservative restoring force f_S, as shown in Fig. 12.

The stiffness of the oscillator is k_1=40,000 [N/m] whenever the absolute value of the displacement is equal or smaller than k_1=0.016 [m]; otherwise, the stiffness is k_2=0.25 k_1. The mass of the oscillator is m=10,000 [kg] and the damping ratio is d=2 % (concerning the stiffness k_1). The force applied over the stiffness is modeled as a discrete white noise of spectral intensity S=10,000 [N^2 s]. The duration of the load is 15 [s] and time is discretized at intervals of 0.01 [s], leading to a total of 1501 random variables for the discrete representation of the stochastic process. The failure event involves the absolute displacement of the SDOF exceeding a threshold δ=0.032 [m].

The problem is solved by the crude MCS, SESC, traditional subset simulation, sequential IS and suggested extrapolation approach for different numbers of support points, and the results are reported in Fig. 13 and Table 4. Results confirm the robustness of the proposed approach for solving this high-dimensional problem noticing the point that the extrapolation by using few support points significantly reduced the function call of SESC (about 45 % and 70 % for 15 and 8 support points, respectively). The performance of the proposed problem is presented in Figs 14 and 15.

Based on the explanations presented in former examples, one may conclude that the proposed approach provides a suitable trade-off between the function call of analysis and accuracy. Ensuring the accuracy of results for a design point-based method requires performing several optimization algorithms to find all design points, which greatly increases the function calls for such high-dimensional problems.

3.2. *Main limitations and potential inefficiencies of the method*

Following the NFLR, to achieve optimal results, the practitioner should choose an approach fit to the structure of the in-hand problem

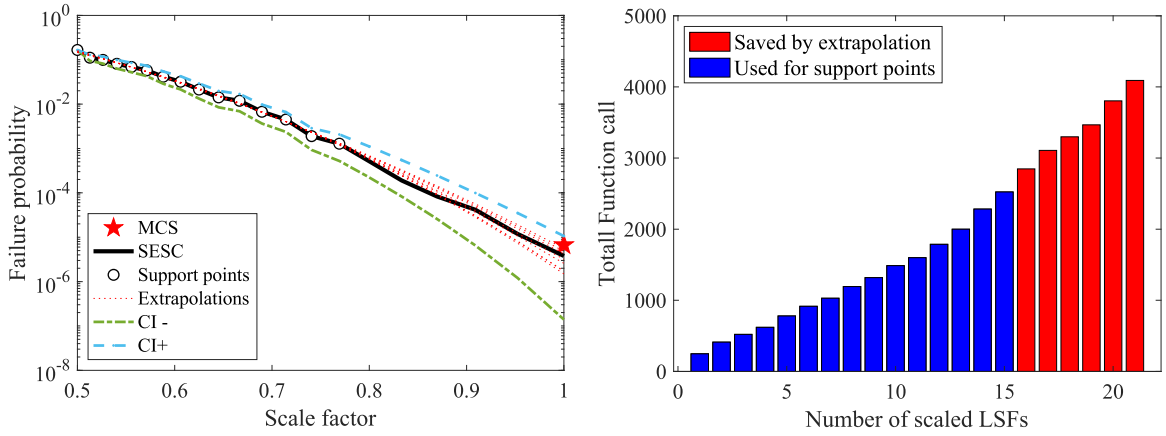

Fig. 14. Results of Asus in solving SDOF reliability problem using 15 support points.

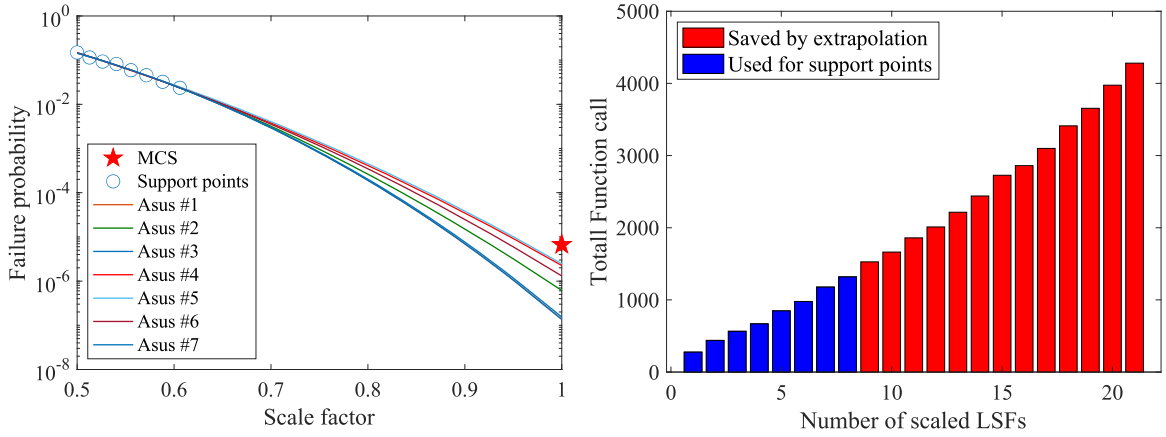

Fig. 15. Results of Asus in solving SDOF reliability problem using 8 support points.

which requires familiarity with the pros and cons of existing reliability methods. Therefore, identifying the limitations of each method would help practitioners make better decisions in reliability analysis. In the case of the proposed framework, the assumptions behind the method raise two main limitations, which we have discussed in detail. Besides, in the case of certain problems, the other extrapolation-based approaches present more efficiency than the proposed approach clarified in this section.

3.2.1. *Problems with high failure probabilities*

The basic assumption of the Gumbel distribution is used to approximate failure probabilities, which is more suitable for extremes and small probabilities (in other words, the method is calibrated only for small probabilities.). In case of problems with high failure probabilities, the marginal function's PDF may not follow the Gumbel distribution, and therefore, the proposed extrapolation cannot be applied to high-probability events.

3.2.2. *Scaling of the Island-shaped failure regions*

In Section 2, to weaken a structural system and explain the concept of the proposed framework, we suggested a specific scaling approach, $g(\lambda \boldsymbol{u})$, which is equal to an increase in the standard deviation of random variables. While this scaling approach can increase the failure probability of many problems, following the NFLR, this approach could not be the optimal solution for all reliability problems. Specifically, for problems with island failure domains, increasing the standard deviation of random variables may paradoxically decrease the failure probability (i.e., $P_{f(\Theta)} \ll P_{f(\Omega)}$), violating the first required condition in Section 2.1.

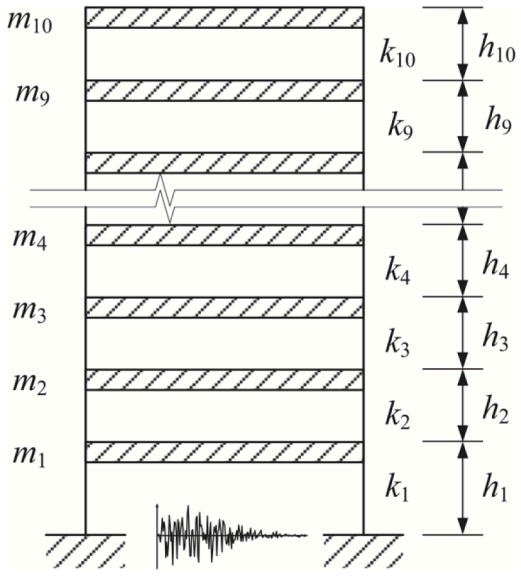

Fig. 16. 10-story nonlinear frame [59].

Therefore, for problems with the Island function, one requires employing another solution to weaken the system. Acknowledging this fact, we provided a list of potential alternative solutions in Table 1 and Proposition 2, which may be used for reliability analysis based on the

Table 5
The random variables of the frame problem.

Parameter	Distribution type	Mean	Coefficient of variation
$m_1 \sim m_3$ (kg)	Lognormal	1.2×10^5	0.1
m_4 (kg)	Lognormal	1.3×10^5	0.1
m_5 (kg)	Lognormal	1.1×10^5	0.1
$m_6 \sim m_7$ (kg)	Lognormal	1.0×10^5	0.1
$m_8 \sim m_9$ (kg)	Lognormal	1.1×10^5	0.1
m_{10} (kg)	Lognormal	0.5×10^5	0.1
E_c (Pa)	Lognormal	3.0×10^{10}	0.1

specification of the in-hand problem.

To examine this issue, we analyze Case 2 of Example 1 in Ref [59]. The problem is a 10-degree-of-freedom lumped-mass nonlinear frame structure subjected to nonstationary ground motions. The schematic representation of the structure and the random variables of the problem are presented in Fig. 16 and Table 5, respectively.

In the case of this problem, the failure regions appear as island-shaped regions, and therefore, increases in the standard deviation of random variables (i.e., scaling LSF as $g(\lambda \boldsymbol{u})$) do not necessarily increase the system's failure probability, presenting a limitation of the proposed scaling approach. Therefore, an alternative scaling method is required for reliability analysis within the proposed asymptotic framework.

To solve the problem, using a few MCMC conditional sampling steps, we employed traditional subset simulation and adapted the scale factor $\lambda_i = \left(\frac{1}{\frac{s_i}{s_1}+1}\right)^{0.25}$ to obtain the exceedance probabilities of the problem (using the solution presented in Proposition 2). For different numbers of support points, the results of the proposed approach are compared with the first four central moment-based SGLD (CM-SGLD) and the conditional extreme value distribution (CEVD) method where the latter employs the first two sampling steps of subset simulation to extrapolate probabilities.

The results of the mentioned approaches are presented in Fig. 17. According to Ref [59], the CEVD and CM-SGLD methods provided a proper approximation of exceedance probabilities using only 950 and 1000 function calls. In the case of the proposed approach, based on the curve fitting technique we have used in our study, at least three/four steps of subset simulation were required (equal to 1400 function calls [59]) to obtain the needed support points for achieving results using Eq. (20).

This example reveals that $g(\lambda \boldsymbol{u})$ fails to properly scale the LSF of these types of functions for extrapolation purposes and alternative scaling

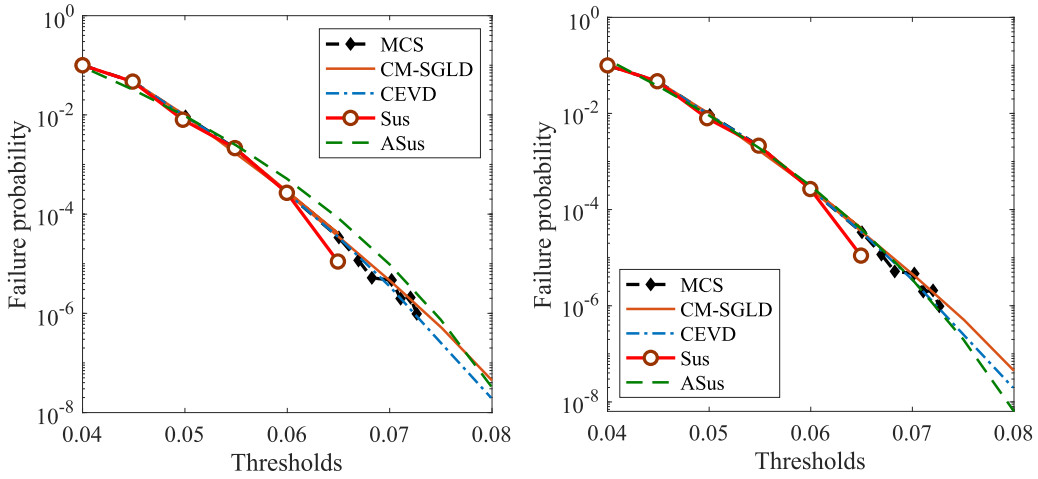

Fig. 17. Exceedance probabilities of the nonlinear frame: Four (Left) and six (Right) steps of traditional subset simulation are used as the support points in the proposed extrapolation approach.

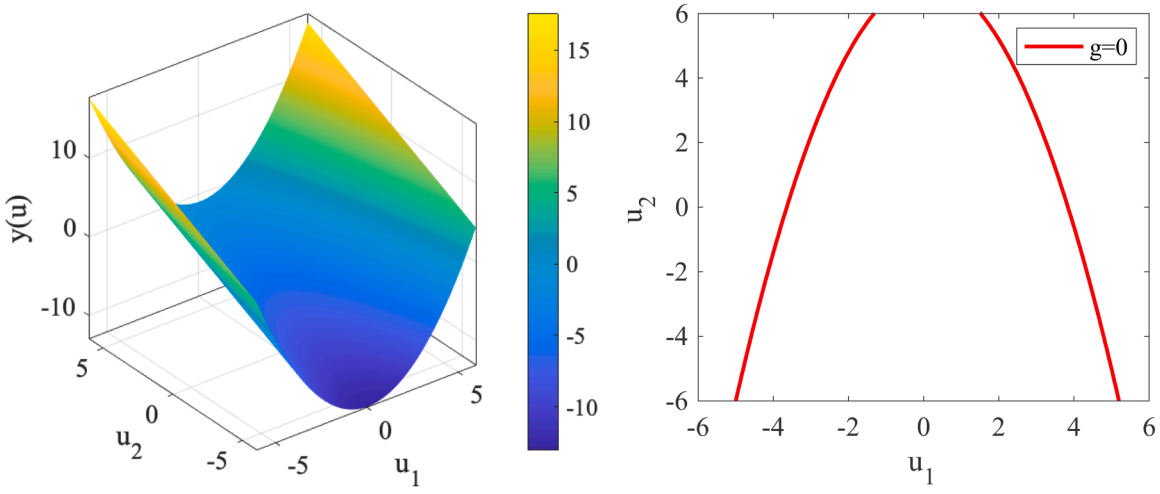

Fig. 18. Example of problems with smooth geometries: For easily predictable problems, alternative algorithms may present more efficiency compared to the proposed approach.

Table 6
The reliability results of the parabola function.

Method	P_f	δ	#g-call
MCS* [55]	3.54×10^{-4}	0.017	10^7
Subset simulation [55]	3.20×10^{-4}	0.184	4312
SESC	1.36×10^{-4}	0.284	4306
DEA [55]	3.75×10^{-4}	0.618	1348
Asus	2.12×10^{-4}	CI=[2.14×10^{-5}, 5.38×10^{-4}]	2352

* Performing MCS with 2,000,000 and 3,000,000 samples, the failure probability obtained by the authors was $P_f = 1.27\times10^{-4}$.

approaches (scale factor of $\lambda_i = \left(\frac{1}{\frac{g_i}{g_1}+1}\right)^{0.25}$ in this case) should be used to solve the problem within the proposed framework. Besides, since the geometry of the performance function is smooth enough, the other extreme values approaches such as the CEVD method (see [59]) present more efficient performance compared to the proposed approach.

3.2.3. *The existence of efficient alternative algorithms for problems with smooth geometries*

The proposed asymptotic approach is designed to explore the failure regions of scaled functions (often requiring over 1000 function calls) to approximate small probabilities. It is well-suited for intricate problems with complex geometries. However, for easily predictable problems

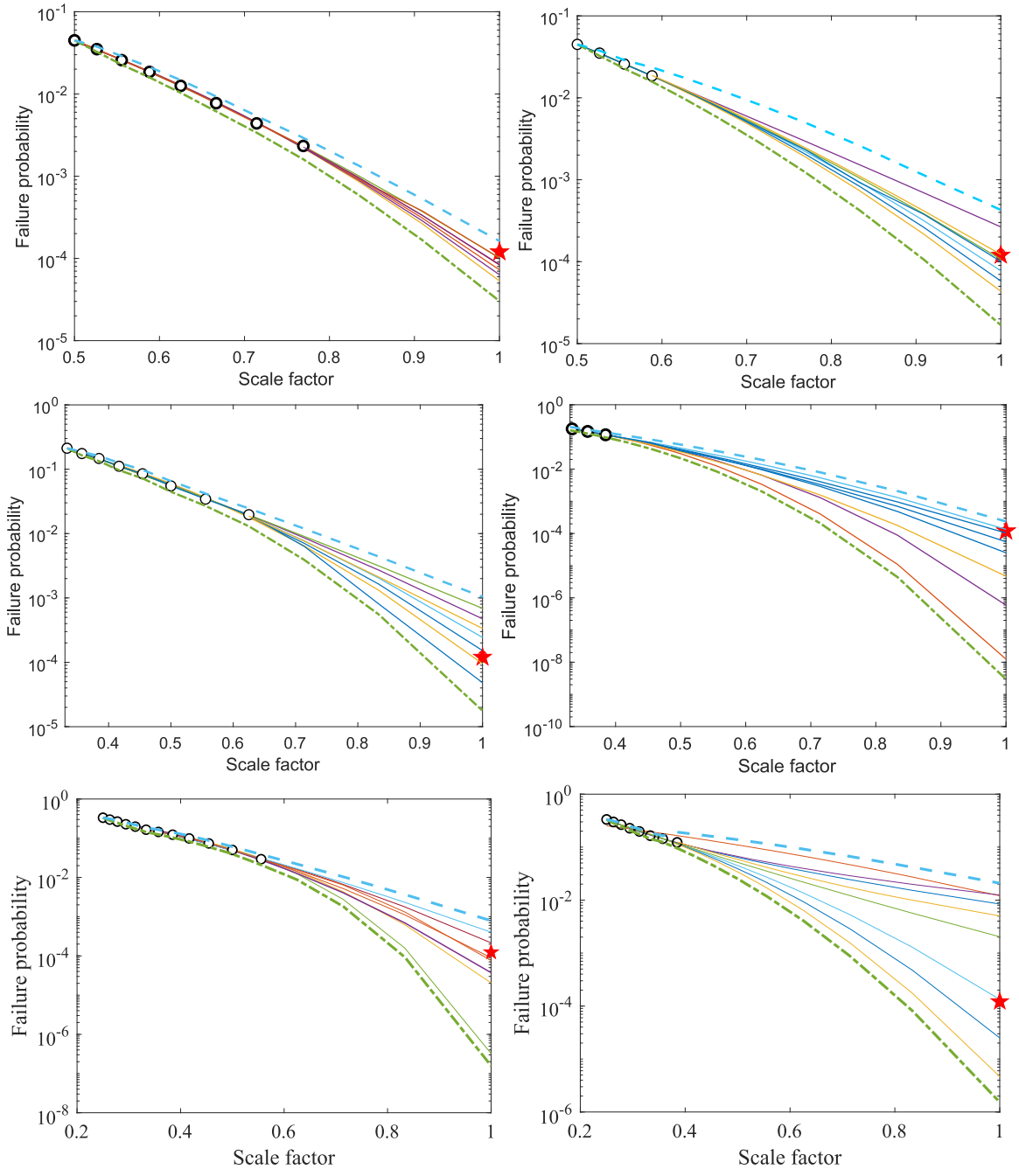

Fig. 19. Influence of the initial scale parameter λ and the number of support points on the confidence interval of failure probabilities.

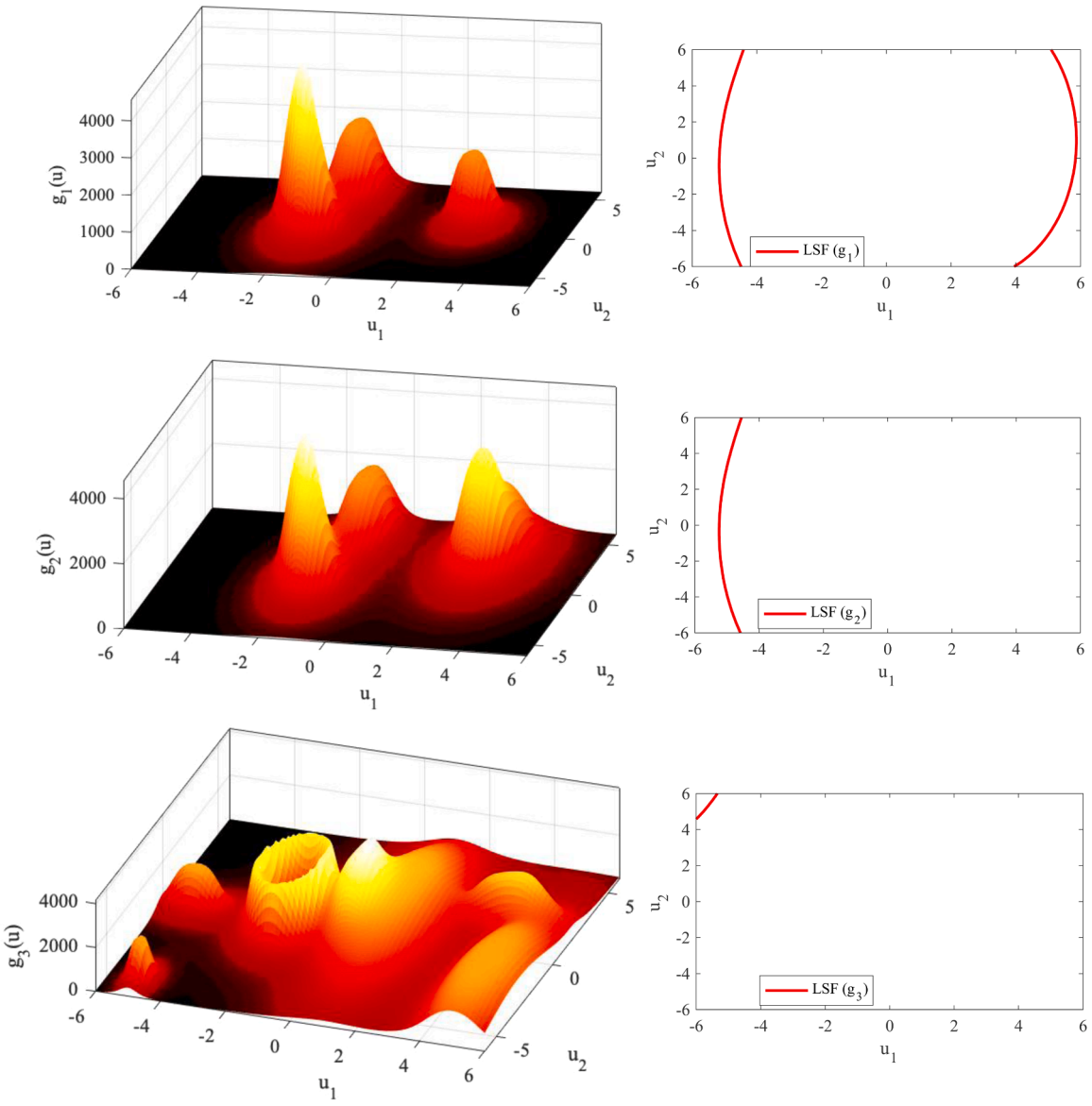

Fig. 20. Example of intricate problems: For such problems with complex geometries, the information obtained from random sampling around the mean may not help traditional approaches to obtain proper information about the failure regions.

with smooth geometries, the proposed approach may be inefficient compared to some state-of-the-art methods, specifically those tailored for such well-behaved functions.

To investigate this issue, a parabola function with standard Gaussian random variables and two important failure regions is investigated [55]:

$$y(\boldsymbol{u}) = -g(u) = u_2 + 0.5(u_1 - 0.1)^2 - 7, \tag{43}$$

System failure is defined as when $y(\boldsymbol{u})$ exceeds the threshold b=0 (see Fig. 18).

To solve such problems, Ref [55] suggested performing a few steps of traditional subset simulation and then extrapolating the PDF of the function using an adaptive density extrapolation approach (DEA). This idea may fail to solve problems with complex geometries (see next subsection), however, as shown in Table 6, it works well for easily predictable problems with smooth geometries. To solve this problem, DEA has used only two steps of traditional subset simulation to solve the problem while the proposed approach employs 8 steps of SESC to approximate the failure probability with an acceptable confidence interval.

To solve the problems, following our experiences and also the conclusion of the former studies [43,46], we used scale parameter $\lambda_1 = 0.5$ in our computations. To provide a more comprehensive understanding of how the parameters of the proposed method influence the confidence interval of failure probabilities, we have conducted a reliability analysis by solving this example for varying numbers of support points and different initial values of the scale parameter, λ_1, within the SESC method. The corresponding results are illustrated in Fig. 19.

As shown in this figure, selecting an initial scale parameter λ_1 that is closer to the one leads to a more desirable confidence interval for the estimated failure probabilities. However, this improvement comes at the expense of increased computational cost, as a finer adjustment of λ_1 reduces the computational effort for probability estimation (i.e., finer λ_1 values would lead to a higher increase in the failure domain). Furthermore, the figure highlights that incorporating a greater number of support points enhances the reliability and robustness of the results. Nevertheless, the selection of an appropriate number of support points is not straightforward and remains a decision that must be carefully made by the practitioner, balancing computational efficiency and the desired level of accuracy in the estimation process.

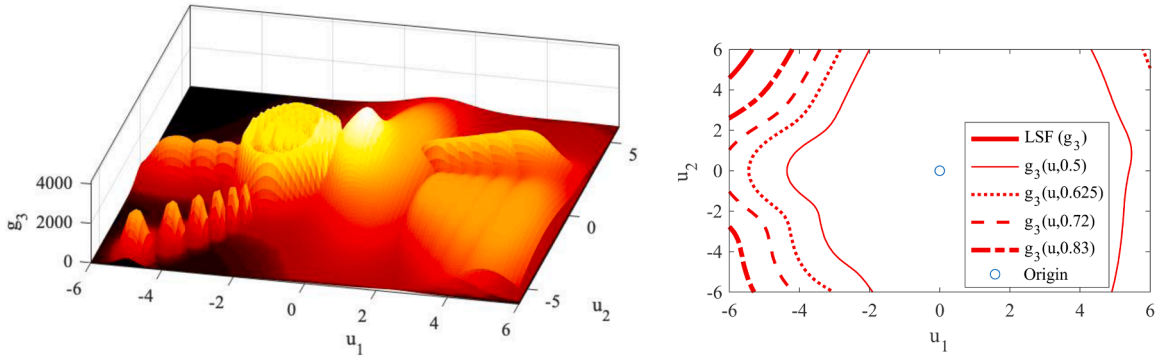

Fig. 21. Scaled versions of an intricate problem (g_3) for probability estimation by the proposed approach.

3.3. Conceptual comparison with the state-of-the-art methods

Generally, the main strategy behind the conditional extrapolation approaches discussed in the literature is generating initial random MCS samples and using few/one conditional MCMC samplings to guess the trend of (the PDF of) the performance function in the next steps and estimate probabilities. However, it is important to consider that this idea only works for certain classes of problems with smooth geometries. A critical question may help us to clarify the issue:

If we have full information about the geometry of a function around the mean, can we use it to extrapolate the geometry far from it?

Generally, except for the smooth functions, the geometry of the performance function near the mean may be completely irrelevant to the regions far from the origin (see discussions in [33,37,53,57]). We have also provided a graphical response to this question in Fig. 20.

In the case of the proposed asymptotic approach, the critical question would be as follows:

If we have failure information of scaled versions of LSFs, can we use it to extrapolate the failure information of the original problem?

In contrast with the state-of-the-art, the extrapolation technique suggested in this study follows the Asymptotic approximation theory which was employed to introduce the SORM [13] method. The Asymptotic theory provides mathematical proofs for the asked critical question and shows that scaled LSF can be used to extrapolate the small failure probability of a problem. For clarification purposes, Fig. 21 shows how function g_3 (presented in Fig. 20) can be scaled to be solved by the proposed approach.

Therefore, in the case of small probabilities estimation, when the application range of the former related studies is restricted to problems with smooth geometries, using a proper scaling approach, the suggested approach can be applied to problems with very complex geometries with high reliability in results.

4. Conclusions

In this study, we followed the asymptotic perceptions for extrapolation of the reliability results of the subset simulations. First, by applying a change in the parameters of the failure probability integral, we suggested presenting a weakened version of a system with a high failure probability that can be estimated by the crude MCS using reasonable function calls. In an equivalent standard normal space, this approach would change the LSF (limit state function) of the problem from $g(\boldsymbol{u})$ to $\widehat{g}(\boldsymbol{u})$. Then, keeping both mentioned LSFs in design space and using the scale parameter λ, we suggested designing a set of nested LSFs between $g(\boldsymbol{u})$ and $\widehat{g}(\boldsymbol{u})$ (with increasing safety domain) in such a manner that for $\lambda = 1$, one obtains $g(\boldsymbol{u}) = \widehat{g}(\boldsymbol{u})$. On one side, we have shown that MCMC sampling and formulation of the SESC (an improved subset simulation) can be used to estimate the failure probability of mentioned nested LSFs (See Eqs. 8 and 10). On the other side, the extreme values theory provides an explicit exponential function for the failure probability of such a problem when λ approaches one (See Eq. 12). By the combination of these perceptions, we have shown that if a few failure probabilities of the nested LSFs are be estimated by former (subset simulation), then, the total failure probability can be extrapolated by the latter (asymptotic approximation in extreme values theory).

The results of this combination are technically important because they reveal the mathematical connections of the subset simulation and asymptotic approaches (e.g., SORM approach) which at first look, seem to not have any connections together. Besides, in case of problems with one important failure region, one may use a few steps of the subset simulation to exploit information about the curvature of the LSF around the design point.

We examined the robustness of the method by solving some intricate problems. First, the Metaball example is used to show the capabilities of the method compared to design point-based methods, traditional subset simulation, and asymptotic sampling.

Then, an engineering problem with 1501 dimensions is solved by the proposed approach. In these cases, reducing the 40 % up to 90 % of function call of the improved subset simulation in the solved examples confirms the potential of the method for application in real-world engineering problems.

Finally, we clarified the main limitations of the proposed approach and studied the potential inefficiency of the proposed approach compared to the state-of-the-art methods. This approach may help practitioners make better decisions in reliability analysis facing related examples.

CRediT authorship contribution statement

Mohsen Rashki: Writing – original draft, Visualization, Methodology, Investigation, Conceptualization. **Matthias G.R. Faes:** Writing – review & editing, Validation, Methodology, Investigation. **Pengfei Wei:** Writing – review & editing, Validation, Methodology, Investigation. **Jingwen Song:** Writing – review & editing, Validation, Methodology, Investigation.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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