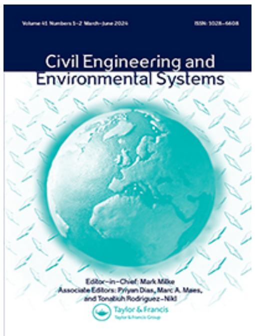

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Marcos A. Valdebenito and Matthias G. R. Faes

Chair for Reliability Engineering, TU Dortmund University, Dortmund, Germany

ABSTRACT
Calculation of failure probabilities and probability sensitivities are very relevant problems in structural reliability. Two of the most widespread methods for estimating such quantities are the First-Order Reliability Method (FORM) and Importance Sampling using design points (IS). This work explores how to employ both approaches in a synergistic way within the framework of Control Variates with Splitting. The results obtained indicate that such a strategy allows obtaining improved estimates of the failure probability and its sensitivity without incurring into additional system analyses. Examples illustrate the application of this framework.

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Failure probability; sensitivity analysis; first-order reliability method; importance sampling; control variates; splitting

1. Introduction

Quantification of the effects of uncertainty in engineering systems has become the subject of active research, as documented in e.g. Beer, Ferson, and Kreinovich (2013) and Faes and Moens (2020). In particular, calculation of the failure probability and its sensitivity are two problems which have received considerable attention in the literature, see e.g. Melchers and Beck (2018). In this context, sensitivity is understood as the partial derivative of the failure probability with respect to either distribution parameters associated with the random variables of a problem or deterministic parameters affecting the system's response, see e.g. Papaioannou, Breitung, and Straub (2013) and Torii and Novotny (2021). In this work, the focus is on the calculation of sensitivity of failure probability with respect to distribution parameters.

Both the failure probability and its sensitivity are quite relevant, as the failure probability expresses the chances that a system undergoes an undesirable behaviour, while the sensitivity reveals the rate of change of that probability with respect to changes in distribution parameters. These two quantities can be calculated resorting to different classes of approaches, such as approximate reliability methods (see e.g. Bjerager and

CONTACT Marcos A. Valdebenito marcos.valdebenito@tu-dortmund.de Chair for Reliability Engineering, TU Dortmund University, Leonhard-Euler-Str. 5, Dortmund 44227, Germany

Krenk 1989; Breitung 1994; Rackwitz and Fiessler 1978), density and path integration methods (see e.g. Kougioumtzoglou, Psaros, and Spanos 2024; Li et al. 2023; Li and Chen 2004), surrogate models (see e.g. Faravelli 1989; Moustapha and Sudret 2019) and simulation techniques (see e.g. Song and Kawai 2023; Wu 1994). Within the class of approximate methods, possibly the most popular approach is the so-called First Order Reliability Method, often abbreviated as FORM. This method heavily relies on linearisation and the concept of the design point and can provide closed-form, analytic expressions for evaluating both the probability (Rackwitz and Fiessler 1978) and its sensitivity (Bjerager and Krenk 1989). Within the class of simulation methods, Monte Carlo and its advanced variants (see e.g. Schuëller, Pradlwarter, and Koutsourelakis 2004) calculate the probability and its sensitivity by generating samples of the uncertain parameters of the problem and evaluating the system's response for each of those realisations. A popular simulation method is Importance Sampling using design points (Melchers 1989; Schuëller and Stix 1987), which is abbreviated in the following as IS. In a nutshell, IS performs simulation around the design point and under certain conditions, can exhibit higher efficiency than Monte Carlo simulation.

A comparison of the performance of FORM and IS indicates that they usually offer a trade-off between precision and accuracy. In this context and according to Menditto, Patriarca, and Magnusson (2007), precision is 'the closeness of agreement between independent test results obtained under stipulated conditions' while accuracy refers the 'closeness of agreement between a quantity value obtained by measurement and the true value of the measurand'. In other words, precision is linked to the concept of variance of an estimate, while accuracy expresses the bias of an estimate, see e.g. Torii (2020). As FORM is based on closed-form analytic expressions that depend on the design point, it produces precise results for probability and its sensitivity. That is, the outcomes of FORM are crisp numbers which approximate the failure probability and its sensitivity. However, the accuracy of FORM (that is, how close FORM estimates are to the true values) is unknown due to the hypothesis of linearisation. This idea is represented schematically in Figure 1, that represents the probability density associated with an estimator of the failure probability (a similar representation would apply to probability sensitivity). As FORM produces a precise estimate, its associated probability density would be actually

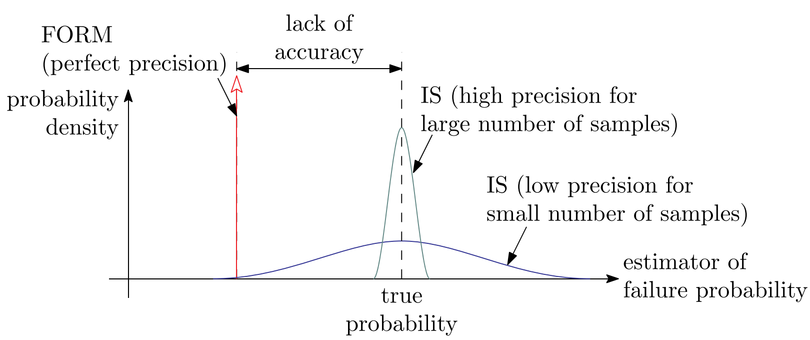

Figure 1. Schematic representation of the concepts of accuracy and precision for estimating failure probability via the First-Order Reliability Method (FORM) and Importance Sampling using design points (IS).

a Dirac delta, which is represented with a red arrow pointing upwards. However, the estimate produced by FORM may be different from the true quantity (the true probability in Figure 1) and hence, it may lack accuracy. In opposition to FORM, IS can offer accurate estimates of probability and its sensitivity, as it does not introduce assumptions about the behaviour of the underlying system (that is, there is no linearisation hypothesis as in FORM). However, the level of precision of those estimates (that is, their variability) will depend (among other factors) on the number of samples considered in the simulation step, see e.g. Torii (2020). Such an idea is represented schematically in Figure 1. In case that IS is applied with a low number of samples as shown with the blue curve, the probability density associated with the estimator will be centred around the true value of the probability meaning that the estimator is – on average – accurate while it exhibits a large variability, implying that it lacks precision. By increasing the number of simulated samples, it is possible to increase the precision of the estimate, as shown with the green curve.

The preceding discussion on the precision of FORM and the accuracy of IS can be broadened by taking into account the numerical costs associated with each reliability method. Indeed, FORM can be quite efficient from a numerical viewpoint for certain classes of problems, as its implementation entails solving an optimisation problem that demands performing few deterministic analyses of the underlying system. In contrast, achieving a good level of precision with IS may demand generating a large number of samples. Generating those samples can become numerically demanding, particularly when the system's performance is evaluated through highly refined numerical models using, e.g. the finite element method (Bathe 1996).

The preceding discussion highlights the advantages and disadvantages of FORM and IS. In such a scenario, this contribution investigates whether it is possible to leverage on FORM to improve the results obtained with IS. Although this is a topic that has been already studied in the past for failure probability estimation (Fujita and Rackwitz 1988; Ghalehnovi, Rashki, and Ameryan 2020), the aim is to shed new light on this issue by resorting to the framework provided by Control Variates with Splitting (CVS), as discussed in Avramidis and Wilson (1993). In a nutshell, Control Variates provides the means for exploiting correlations between two or more estimators to produce improved estimators with minimal variance and hence, a high level of precision (see e.g. Fishman 1996; Ng and Willcox 2014; Peherstorfer, Willcox, and Gunzburger 2018). The Splitting approach is helpful for setting the optimal control parameters associated with the implementation of Control Variates and consists of generating estimators of the sought quantities based on subsets of realisations. Therefore, the framework provided by CVS benefits from the precision associated with the estimates produced by FORM and the accuracy of the estimates associated with IS. Moreover, it is shown that the framework provided by CVS allows estimating both the failure probability and its sensitivity. The scope of application of the framework considered in this paper involves problems of structural reliability that exhibit a linear or moderately non-linear behaviour with respect to the uncertain input parameters, while the number of uncertain inputs itself is also moderate (Katafygiotis and Zuev 2008). For such problems, it is expected that the linearisation hypothesis associated with FORM is representative with respect to the actual geometry of a reliability problem.

The rest of this contribution is organised as follows. Section 2 provides definitions of the failure probability and its sensitivity, which are the quantities of interest of this work. Then, Section 3 reviews the basics of FORM and IS for probability and sensitivity estimation. As both of these methods are well known, only very essential concepts which are required for this contribution are reviewed. The integration of FORM and IS within the framework of CVS is discussed in Section 4 and its practical application is illustrated in Section 5. Conclusions are drawn in Section 6.

2. Definition of the problem

Consider a structure or system whose behaviour is characterised through a numerical model using, e.g. the finite element method (Bathe 1996). The numerical model possesses n input variables x_i, $i = 1, \ldots, n$ that describe, e.g. loading, material properties, etc. These input parameters are affected by uncertainty, which is described by means of independent random variables with probability density function $f_{X_i}(x_i \mid \boldsymbol{\theta}_i)$, where $\boldsymbol{\theta}_i$ is a vector that collects the distribution parameters of X_i such as mean, standard deviation, etc. In view of the assumption of independence, the joint probability density function is $f_{\boldsymbol{X}}(\boldsymbol{x} \mid \boldsymbol{\theta}) = \prod_{i=1}^{n} f_{X_i}(x_i \mid \boldsymbol{\theta}_i)$, where $\boldsymbol{x} = [x_1, \ldots, x_n]^T$, $\boldsymbol{\theta} = [\boldsymbol{\theta}_1^T, \ldots, \boldsymbol{\theta}_n^T]^T$ and $(\cdot)^T$ represents transpose.

The behaviour of the system is synthesised in the so-called performance function $g_{\boldsymbol{x}}(\boldsymbol{x})$, which assumes a value equal or smaller than zero whenever a combination of the uncertain input parameters $\boldsymbol{x}$ leads to an undesirable response, for example, loss of serviceability or collapse (see e.g. Bucher 2009; Ditlevsen and Madsen 1996). Thus, the chances that the system undergoes an undesirable behaviour are given by the classical probability integral (Melchers and Beck 2018):

$$p_F = \int_{\boldsymbol{x} \in \Omega_{\boldsymbol{x}}} I_{\boldsymbol{x}}(\boldsymbol{x}) f_{\boldsymbol{X}}(\boldsymbol{x} \mid \boldsymbol{\theta}) \, \mathrm{d}\boldsymbol{x}, \tag{1}$$

where p_F is the failure probability, $\Omega_{\boldsymbol{x}}$ is the domain associated with $\boldsymbol{x}$ and $I_{\boldsymbol{x}}(\boldsymbol{x})$ is the indicator function (Ditlevsen and Madsen 1996). The indicator function is such that $I_{\boldsymbol{x}}(\boldsymbol{x}) = 1$ whenever $g_{\boldsymbol{x}}(\boldsymbol{x}) \leq 0$, and $I_{\boldsymbol{x}}(\boldsymbol{x}) = 0$ otherwise. An inspection of Equation (1) reveals that the failure probability depends on the vector of distribution parameters $\boldsymbol{\theta}$. Thus, the partial derivative of p_F with respect to distribution parameters provides useful information on its sensitivity (Wu 1994):

$$\frac{\partial p_F}{\partial \theta_{l,i}} = \int_{\boldsymbol{x} \in \Omega_{\boldsymbol{x}}} I_{\boldsymbol{x}}(\boldsymbol{x}) \eta_{x_i,\theta_{l,i}}(x_i \mid \boldsymbol{\theta}_i) f_{\boldsymbol{X}}(\boldsymbol{x} \mid \boldsymbol{\theta}) \, \mathrm{d}\boldsymbol{x}, \quad l = 1, \ldots, n_i, \; i = 1, \ldots, n, \tag{2}$$

where $\theta_{l,i}$ represents the l-th distribution parameter associated with the i-th random variable, n_i is the number of distribution parameters associated with the i-th random variable and $\eta_{x_i,\theta_{l,i}}(x_i \mid \boldsymbol{\theta}_i)$ is the so-called score function (Rubinstein and Kroese 2016):

$$\eta_{x_i,\theta_{l,i}}(x_i \mid \boldsymbol{\theta}_i) = \frac{1}{f_{X_i}(x_i \mid \boldsymbol{\theta}_i)} \frac{\partial f_{X_i}(x_i \mid \boldsymbol{\theta}_i)}{\partial \theta_{l,i}}. \tag{3}$$

Note that the score function can be calculated in closed form for several different types of distributions, as discussed in e.g. Rubinstein and Kroese (2016).

The calculation of the failure probability p_F and its sensitivity $\partial p_F/\partial\theta_{l,i}$ constitute two of the central problems of structural reliability theory. For cases of practical interest, such calculation is challenging, as the performance function $g_{\boldsymbol{x}}(\boldsymbol{x})$ is not known analytically. Instead, it can be evaluated point-wise for specific realisations of the vector of input parameters $\boldsymbol{x}$ (Schuëller, Pradlwarter, and Koutsourelakis 2004). Therefore, specialised methods for its evaluation are required, as discussed in the following.

3. Basic aspects of the first-order reliability method and importance sampling using design points

3.1. Transformation into standard normal space

Before discussing methods such as the First-Order Reliability Method (FORM) and Importance Sampling using design points (IS), it is useful to discuss the representation of the probability and sensitivity integrals into the standard normal space. Such is a usual step when performing reliability analysis (Der Kiureghian 2004) and consists of applying an iso-probabilistic transformation that relates the so-called physical space (associated with the random variables X_i, $i = 1, \ldots, n$ of the reliability problem) with the standard normal space. Different types transformations can be considered, for example, Rosenblatt's of Nataf's transformation (Ditlevsen and Madsen 1996; Liu and Der Kiureghian 1986). In view of the assumption of independence between random variables, the mapping between physical input variables x_i, $i = 1, \ldots, n$, and their standard normal counterparts z_i, $1, \ldots, n$, is expressed as $z_i = t_i(x_i \mid \boldsymbol{\theta}_i)$, $i = 1, \ldots, n$. The function $t_i(x_i \mid \boldsymbol{\theta}_i)$ depends on the associated cumulative distribution function of X_i and the inverse cumulative distribution of a standard normal random variable (Der Kiureghian 2004). The vector-valued counterpart of the aforementioned mapping function is denoted as $\boldsymbol{z} = \boldsymbol{t}(\boldsymbol{x} \mid \boldsymbol{\theta})$ and its inverse is denoted as $\boldsymbol{x} = \boldsymbol{t}^{-1}(\boldsymbol{z} \mid \boldsymbol{\theta})$.

Considering the definitions discussed above and applying change of variables over Equations (1) and (2), the failure probability integral and its sensitivity are recast as:

$$p_F = \int_{\boldsymbol{z}\in\mathcal{R}^n} I_{\boldsymbol{z}}(\boldsymbol{z})\phi_n(\boldsymbol{z})\,\mathrm{d}\boldsymbol{z}, \tag{4}$$

$$\frac{\partial p_F}{\partial\theta_{l,i}} = \int_{\boldsymbol{z}\in\mathcal{R}^n} I_{\boldsymbol{z}}(\boldsymbol{z})\eta_{z_i,\theta_{l,i}}(z_i \mid \boldsymbol{\theta}_i)\phi_n(\boldsymbol{z})\,\mathrm{d}\boldsymbol{z}. \tag{5}$$

In the above equations, note that $I_{\boldsymbol{z}}(\boldsymbol{z}) = I_{\boldsymbol{x}}(\boldsymbol{t}^{-1}(\boldsymbol{z} \mid \boldsymbol{\theta}))$, $\eta_{z_i,\theta_{l,i}}(z_i \mid \boldsymbol{\theta}_i) = \eta_{x_i,\theta_{l,i}}(t_i^{-1}(z_i \mid \boldsymbol{\theta}_i) \mid \boldsymbol{\theta}_i)$ and $\phi_n(\cdot)$ denotes the n-dimensional independent standard normal probability density function. In addition, $t_i^{-1}(\cdot)$ represents the i-th component of $\boldsymbol{t}^{-1}(\cdot)$ and $\mathcal{R}$ denotes the set of real numbers.

3.2. First-order reliability method (FORM)

The First-Order Reliability Method (FORM) estimates the probability integral by approximating the limit state surface $g_{\boldsymbol{z}}(\boldsymbol{z})$ around the so-called design point $\boldsymbol{z}^*$ by means of a hyperplane. In this context, note that the design point is the realisation in the standard normal space which possesses smallest Euclidean norm with respect to the origin such that it lies within the failure domain, that is, $g_{\boldsymbol{z}}(\boldsymbol{z}) \leq 0$. The identification of the design

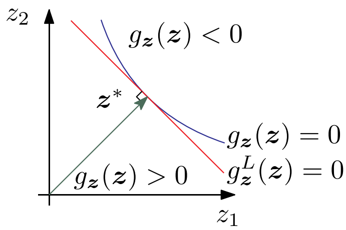

Figure 2. Schematic representation of the limit state surface $g_{\boldsymbol{z}}(\boldsymbol{z}) = 0$, the design point $\boldsymbol{z}^*$ and the approximation of the limit state surface according to FORM $g^L_{\boldsymbol{z}}(\boldsymbol{z}) = 0$.

point entails solving an optimisation problem, which can be carried out using any appropriate numerical scheme, see e.g. Der Kiureghian (2004). A schematic representation of both the design point $\boldsymbol{z}^*$ and the hyperplane $g^L_{\boldsymbol{z}}(\boldsymbol{z}) = 0$ that approximates the limit state surface according to the FORM hypothesis is depicted in Figure 2. The hyperplane approximating the performance function allows approximating the indicator function $I_{\boldsymbol{z}}(\boldsymbol{z})$ in Equations (4) and (5) as:

$$I_{\boldsymbol{z}}(\boldsymbol{z}) \approx \boldsymbol{Iz}^{\text{FORM}}(\boldsymbol{z}) = \begin{cases} \boldsymbol{1} & \text{if } \alpha^{\boldsymbol{T}}\boldsymbol{z} \geq \beta \\ \boldsymbol{0} & \text{otherwise} \end{cases} \tag{6}$$

where β is the so-called reliability index, which is the Euclidean norm of the design point, that is $\beta = ||\boldsymbol{z}^*||_2$; and α is a unit vector pointing towards the design point, that is $\alpha = \boldsymbol{z}^*/||\boldsymbol{z}^*||_2$. Considering such an approximation, the estimates for the failure probability and its sensitivity according to FORM become (Bjerager and Krenk 1989; Rackwitz and Fiessler 1978):

$$p_F \approx p_F^{\text{FORM}} = \int_{\boldsymbol{z}\in\mathcal{R}^n} I_{\boldsymbol{z}}^{\text{FORM}}(\boldsymbol{z})\phi_n(\boldsymbol{z})\,\mathrm{d}\boldsymbol{z} = \Phi(-\beta), \tag{7}$$

$$\frac{\partial p_F}{\partial \theta_{l,i}} \approx \frac{\partial p_F^{\text{FORM}}}{\partial \theta_{l,i}} = \int_{\boldsymbol{z}\in\mathcal{R}^n} I_{\boldsymbol{z}}^{\text{FORM}}(\boldsymbol{z})\eta_{z_i,\theta_{l,i}}(z_i \mid \boldsymbol{\theta}_i)\phi_n(\boldsymbol{z})\,\mathrm{d}\boldsymbol{z} = -\phi(-\beta)\alpha_i\frac{\partial z_i^*}{\partial \theta_{l,i}}, \tag{8}$$

where $\phi(\cdot)$ and $\Phi(\cdot)$ denote the standard normal probability density and cumulative distribution functions, respectively; and z_i^* is the i-th component of the design point $\boldsymbol{z}^*$. Note that the estimates for the probability and its sensitivity according to FORM as presented in Equations (7) and (8), respectively, are actually closed-form expressions that depend on the information associated with the design point (see, e.g. Breitung 2024).

3.3. Importance sampling using design points (IS)

Importance Sampling using design points (IS) consists of introducing a sampling density function centred at the design point (Melchers 1989; Schuëller and Stix 1987). Such density allows drawing realisations of the uncertain variables that lie into the failure domain more frequently than with, e.g. Monte Carlo simulation. A straightforward choice for the importance sampling density function is a multivariate normal distribution with mean equal to the design point and unit covariance matrix (Au and Beck 2003). Thus, the expressions for estimating the probability and its sensitivity considering IS become (Melchers 1989; Rubinstein and Kroese 2016; Schuëller and Stix 1987; Wu 1994):

$$p_F = \int_{\mathbf{z}\in\mathcal{R}^n} I_{\mathbf{z}}(\mathbf{z}) \frac{\phi_n(\mathbf{z})}{\phi_n(\mathbf{z}\,|\,\mathbf{z}^*)} \phi_n(\mathbf{z}\,|\,\mathbf{z}^*)\mathrm{d}\mathbf{z}, \tag{9}$$

$$\frac{\partial p_F}{\partial \theta_{l,i}} = \int_{\mathbf{z}\in\mathcal{R}^n} I_{\mathbf{z}}(\mathbf{z}) \eta_{z_i,\theta_{l,i}}(z_i \mid \boldsymbol{\theta}_i) \frac{\phi_n(\mathbf{z})}{\phi_n(\mathbf{z}\,|\,\mathbf{z}^*)} \phi_n(\mathbf{z}\,|\,\mathbf{z}^*)\mathrm{d}\mathbf{z}, \tag{10}$$

where $\phi_n(\mathbf{z}\,|\,\mathbf{z}^*)$ denotes a multivariate normal distribution with mean equal to the design point and unit covariance matrix. These expressions are approximated by generating a a sample of size N, leading to:

$$p_F \approx \widehat{p}_F^{\mathrm{IS}}(\mathbf{Z}_N) = \frac{1}{N}\sum_{j=1}^{N} I_{\mathbf{z}}\left(\mathbf{z}^{(j)}\right) \frac{\phi_n\left(\mathbf{z}^{(j)}\right)}{\phi_n\left(\mathbf{z}^{(j)}\,|\,\mathbf{z}^*\right)}, \tag{11}$$

$$\frac{\partial p_F}{\partial \theta_{l,i}} \approx \frac{\partial \widehat{p}_F^{\mathrm{IS}}}{\partial \theta_{l,i}}(\mathbf{Z}_N) = \frac{1}{N}\sum_{j=1}^{N} I_{\mathbf{z}}\left(\mathbf{z}^{(j)}\right) \eta_{z_i,\theta_{l,i}}\left(z_i^{(j)} \mid \boldsymbol{\theta}_i\right) \frac{\phi_n\left(\mathbf{z}^{(j)}\right)}{\phi_n\left(\mathbf{z}^{(j)}\,|\,\mathbf{z}^*\right)}, \tag{12}$$

where $\mathbf{z}^{(j)}$, $j = 1, \ldots, N$ are independent and identically distributed samples drawn according to $\phi_n(\mathbf{z}\,|\,\mathbf{z}^*)$ and $\mathbf{Z}_N$ is a matrix of dimension $n \times N$ such that each of its columns contains one sample vector $\mathbf{z}^{(j)}$, $j = 1, \ldots, N$.

The expressions in Equations (11) and (12) provide the mean estimates for the failure probability and its sensitivity, respectively. In addition, it is also possible to calculate the variance associated with those estimates, see e.g. Melchers (1989), Rubinstein and Kroese (2016) and Schuëller and Stix (1987) and also the Appendix.

Please note that the above description of Importance Sampling is a specific one that involves design points. However, more general versions of Importance Sampling do not necessarily resort to the concept of design point, as documented in e.g. Papaioannou, Papadimitriou, and Straub (2016). Nevertheless, this work focuses exclusively on the variant of Importance Sampling considering design points.

4. Aggregating the results of the first-order reliability method and importance sampling using design points via control variates with splitting

4.1. Preliminary remarks

The description of FORM and IS as described in Section 3 shows that both reliability methods are applied in sequence. That is, the application of IS as described in Section 3.3 demands knowledge on the design point associated with FORM as discussed in Section 3.2. Such an issue has been acknowledged in the literature, see e.g. Fujita and Rackwitz (1988). In this section, the relationship between FORM and IS is exploited using the framework of Control Variates (see e.g. Fishman 1996). The focus is first on examining the application of Control Variates for the estimation of the failure probability, as discussed in Sections 4.2 and 4.3. Then, this framework is extended towards probability sensitivity estimation, as discussed in Section 4.4.

4.2. Estimation of the failure probability using control variates

Before applying the framework of Control Variates, it is necessary to introduce the following definition. Consider the expression for estimating the failure probability by means of IS as shown in Equation (11). This expression is re-written by considering the approximate indicator function associated with FORM, leading to:

$$p_F^{\text{FORM}} \approx \widehat{p}_F^{\text{FORM}}(\boldsymbol{Z}_N) = \frac{1}{N}\sum_{j=1}^{N} I_{\boldsymbol{z}}^{\text{FORM}}\left(\boldsymbol{z}^{(j)}\right)\frac{\phi_n\left(\boldsymbol{z}^{(j)}\right)}{\phi_n\left(\boldsymbol{z}^{(j)} \mid \boldsymbol{z}^*\right)}. \tag{13}$$

This last expression could be interpreted as an estimator of the failure probability associated with the FORM hypothesis calculated using IS. Clearly, calculating this expression is not practical, as there is a closed-form expression for the probability integral under the FORM hypothesis, as shown in Equation (7). However, the seemingly useless expression in Equation (13) is required in Control Variates, as discussed in the following.

The expression for estimating the failure probability within the context of Control Variates (abbreviated as CV) is, see e.g. Fishman (1996) and Ng and Willcox (2014):

$$\widehat{p}_F^{\text{CV}} = \widehat{p}_F^{\text{IS}}(\boldsymbol{Z}_n) - \gamma\widehat{p}_F^{\text{FORM}}(\boldsymbol{Z}_N) + \gamma p_F^{\text{FORM}}, \tag{14}$$

where $\widehat{p}_F^{\text{FORM}}(\boldsymbol{Z}_N)$ has been defined in Equation (13) and γ is real number denoted as the *control parameter*. The precise role of this control parameter is discussed later on.

An examination of the structure of Equation (14) indicates that:

- The term γp_F^{FORM} provides a precise (that is, a crisp number) but eventually not accurate estimate of the failure probability (that is, different from the true probability value), as it is calculated based on the FORM hypothesis.
- The term $\widehat{p}_F^{\text{IS}}(\boldsymbol{Z}_n) - \gamma\widehat{p}_F^{\text{FORM}}(\boldsymbol{Z}_N)$ acts as a correction of the previous term. Note that the estimates $\widehat{p}_F^{\text{IS}}$ and $\widehat{p}_F^{\text{FORM}}$ are evaluated considering the same set of samples $\boldsymbol{Z}_N$. This is a key issue within Control Variates, as this allows exploiting correlations between both estimates (Fishman 1996).

A salient feature of the CV estimate as shown in Equation (14) is that it produces accurate estimates of the failure probability for an arbitrary value of the control parameter γ (Fishman 1996; Ng and Willcox 2014). The latter holds even though Equation (14) involves the FORM estimate. This is easily verified by noting that the expected value of the terms $-\gamma\widehat{p}_F^{\text{FORM}}(\boldsymbol{Z}_N) + \gamma p_F^{\text{FORM}}$ is equal to zero.

The discussion above sheds some light on the main characteristics of the Control Variates approach as described in Equation (14), although it does not discuss the role of the control parameter γ. Such role can be understood as follows. For those cases where hypothesis associated with FORM (that is, linearisation of the limit state surface at the design point) is representative of the actual limit state surface, the probability estimate provided by FORM is not only precise but also accurate. This implies that such estimate can be given much confidence, which is reflected by a control parameter tending to one, that is $\gamma \to 1$. In case that the opposite holds (that is, the FORM hypothesis is not representative), more importance should be given to the probability estimate generated with IS. In turn, this implies that the control parameter is such that $\gamma \to 0$. Of course, determining the quality of the FORM hypothesis may not be straightforward, opening the question about how the control parameter γ is selected in practice. Such an issue has been addressed in the literature by minimising the variance associated with the CV estimate in Equation (14), which means that the estimator is forced to be as accurate as possible. To achieve this goal, consider the estimate of the variance of the estimator in Equation (14) is (Fishman 1996):

$$\widehat{\sigma}^2\left[\widehat{p}_F^{\mathrm{CV}}\right] = \widehat{\sigma}^2\left[\widehat{p}_F^{\mathrm{IS}}(\boldsymbol{Z}_N)\right] - 2\gamma\widehat{\delta}\left[\widehat{p}_F^{\mathrm{IS}}(\boldsymbol{Z}_N),\ \widehat{p}_F^{\mathrm{FORM}}(\boldsymbol{Z}_N)\right] + \gamma^2\widehat{\sigma}^2\left[\widehat{p}_F^{\mathrm{FORM}}(\boldsymbol{Z}_N)\right] \tag{15}$$

where $\widehat{\sigma}^2[\,\cdot\,]$ and $\widehat{\delta}[\,\cdot\,,\ \cdot\,]$ are the variance and covariance estimators of the corresponding arguments. Please note that detailed expressions for estimating the variances and covariance involved in Equation (15) can be found in the Appendix. As this expression is quadratic with respect to the control parameter γ, it is possible to select this parameter such that it minimises the variance estimator in Equation (15). By imposing that the derivative of Equation (15) with respect to γ is equal to zero, one finds the optimal control parameter γ^* (Fishman 1996; Nelson 1989).

$$\gamma^* = \frac{\widehat{\delta}\left[\widehat{p}_F^{\mathrm{IS}}(\boldsymbol{Z}_N),\ \widehat{p}_F^{\mathrm{FORM}}(\boldsymbol{Z}_N)\right]}{\widehat{\sigma}^2\left[\widehat{p}_F^{\mathrm{FORM}}(\boldsymbol{Z}_N)\right]}. \tag{16}$$

Introducing the optimal control parameter γ^* in Equation (16) into Equation (15) yields the minimum possible variance of the estimator of the failure probability following the Control Variates scheme. Such minimum variance is the following (Fishman 1996; Nelson 1989).

$$\widehat{\sigma}^2_{\min}\left[\widehat{p}_F^{\mathrm{CV}}\right] = \widehat{\sigma}^2\left[\widehat{p}_F^{\mathrm{IS}}(\boldsymbol{Z}_N)\right] - \frac{\left(\widehat{\delta}\left[\widehat{p}_F^{\mathrm{IS}}(\boldsymbol{Z}_N),\ \widehat{p}_F^{\mathrm{FORM}}(\boldsymbol{Z}_N)\right]\right)^2}{\widehat{\sigma}^2\left[\widehat{p}_F^{\mathrm{FORM}}(\boldsymbol{Z}_N)\right]} \tag{17}$$

From this last expression, it is interesting to discuss the following two extreme cases.

- In case that the covariance between $\widehat{p}_F^{\mathrm{IS}}$ and $\widehat{p}_F^{\mathrm{FORM}}$ is high, the minimum variance tends to zero. Such behaviour can be understood as follows. A high covariance indicates that the FORM hypothesis is actually quite good. This implies that the actual limit state surface can be reasonably approximated by a hyperplane at the design point. Therefore, the estimate of the failure probability provided by FORM is not only precise but also accurate. In this situation $\gamma^* \to 1$, confirming that the FORM estimate in Equation (14) receives a high relative weight. This represents an ideal case, as the variance of the Control Variates estimate becomes very small.
- In case that the covariance between $\widehat{p}_F^{\mathrm{IS}}$ and $\widehat{p}_F^{\mathrm{FORM}}$ is close to zero, the minimum variance tends to the variance of the estimator associated with $\widehat{p}_F^{\mathrm{IS}}$. In this case, a low covariance indicates that the FORM hypothesis is not appropriate. This means that considering a hyperplane about the design point to approximate the true limit state surface is not representative. Therefore, the estimate of the failure probability provided by FORM does not provide useful information and hence, its information is not relevant. Indeed, in such a situation, the optimal control parameter $\gamma^* \to 0$, meaning that the FORM approximation is practically discarded from the Control Variates

expression as shown in Equation (14). Hence, applying Control Variates does not report a significant advantage.

The above discussion highlights that the role of the optimal control parameter is weighting the relative importance of the FORM approximation and that associated with IS. Moreover, this parameter is evaluated through Equation (16), which depends only on the information generated when applying FORM and IS. Thus, it is noted that actually Control Variates can be interpreted as a post-processing step of FORM and IS. This is most attractive from a practical viewpoint, as the implementation of Control Variates does not entail additional evaluations of the performance function and hence, no additional system analyses using, e.g. the finite element method.

4.3. Estimation of the failure probability using control variates with splitting

The importance of the optimal control parameter γ^* is evident from the previous Section. Moreover, Equation (16) provides a concrete means for choosing this parameter. However, if the same samples $\boldsymbol{Z}_N$ are used to evaluate both the probability estimators and the optimal control parameter through the aforementioned equation, the resulting Control Variates estimate becomes biased (Pasupathy et al. 2012). The source of this bias has been examined in, e.g. Nelson (1990), and it can be understood as follows. In Section 4.2, it is indicated that Control Variates produces accurate estimates as long as the control parameter γ is arbitrary. However, by applying Equation (16), the control parameter is no longer arbitrary and in fact, it will be correlated with the different quantities being estimated, as they all depend on the sample set $\boldsymbol{Z}_N$. A simple means to overcome this issue is to apply the Splitting technique proposed by Avramidis and Wilson (1993). Such technique involves generating $N_S \geq 3$ subsets out of the sample set $\boldsymbol{Z}_N$. These subsets are denoted as $\boldsymbol{Z}_{N^*}^{(k)}$, $k = 1, \ldots, N_S$, where each set contains a total of $N^* = N/N_S$ elements. It is implicitly assumed that N^* is an integer number, such that samples contained in $\boldsymbol{Z}_N$ are evenly distributed among the N_S subsets. Taking into account these subsets of samples, the estimator of the failure probability considering Control Variates with Splitting (abbreviated as CVS) is, see e.g. Avramidis and Wilson (1993):

$$\begin{aligned} \widehat{p}_F^{\text{CVS}} = \frac{1}{N_S}\sum_{k=1}^{N_S}\Big(&\widehat{p}_F^{\text{IS}}\big(\boldsymbol{Z}_{N^*}^{(k)}\big) - \gamma^*\big(\boldsymbol{Z}_{N^*}^{\tau(k)}\big)\widehat{p}_F^{\text{FORM}}\big(\boldsymbol{Z}_{N^*}^{(k)}\big) \\ &+\gamma^*\big(\boldsymbol{Z}_{N^*}^{\tau(k)}\big)p_F^{\text{FORM}}\Big), \end{aligned} \tag{18}$$

where $\widehat{p}_F^{\text{IS}}(\boldsymbol{Z}_{N^*}^{(k)})$ and $\widehat{p}_F^{\text{FORM}}(\boldsymbol{Z}_{N^*}^{(k)})$ are the estimators in Equations (11) and (13) evaluated with the sample subset $\boldsymbol{Z}_{N^*}^{(k)}$; $\gamma^*(\boldsymbol{Z}_{N^*}^{\tau(k)})$ is the optimal control parameter associated with the sample subset $\boldsymbol{Z}_{N^*}^{\tau(k)}$, that is:

$$\gamma^*\big(\boldsymbol{Z}_{N^*}^{\tau(k)}\big) = \frac{\widehat{\delta}\big[\widehat{p}_F^{\text{IS}}\big(\boldsymbol{Z}_N^{\tau(k)}\big), \widehat{p}_F^{\text{FORM}}\big(\boldsymbol{Z}_N^{\tau(k)}\big)\big]}{\widehat{\sigma}^2\big[\widehat{p}_F^{\text{FORM}}\big(\boldsymbol{Z}_N^{\tau(k)}\big)\big]} \tag{19}$$

and $\tau(k)$ is the so-called subset controller, which is actually an integer that allows identifying different subsets and is defined as, see e.g. Avramidis and Wilson (1993):

$$\tau(k) = \text{mod}(k, N_S) + 1, \quad k = 1, \ldots, N_S \tag{20}$$

where $\text{mod}(k, N_S)$ returns the remainder of the division between k and N_S. Note that whenever $N_S \geq 2$, it is ensured that $\tau(k) \neq k$, $k = 1, \ldots, N_S$. Equation (18) can be interpreted as follows. It provides an average over N_S estimators, where for each individual estimator, the quantities $\widehat{p}_F^{\text{IS}}$ and $\widehat{p}_F^{\text{FORM}}$ are estimated with the sample subset $\boldsymbol{Z}_{N^*}^{(k)}$ while the optimal control parameter γ^* is estimated with the sample subset $\boldsymbol{Z}_{N^*}^{\tau(k)}$. Such a strategy effectively removes the effect of bias, as demonstrated by Avramidis and Wilson (1993), as the estimators of the failure probability are *controlled* using samples from a different subset. In other words, correlations between the estimated failure probabilities and the control parameters are effectively broken because they are calculated with subsets containing different samples. While in principle $N_S \geq 3$, in practice the number of subsets is selected as $N_S = 3$. In that way, the effect of additional variance due to the presence of the subsets is minimised.

The whole concept behind the Splitting scheme is depicted schematically in Figure 3. In this Figure, the sample set contains a total of $N = 9$ samples, where each individual sample is denoted with a red dot. These samples are split into $N_S = 3$ subsets, each containing $N^* = 3$ samples. Then, the samples contained in the first subset are used to estimate $\widehat{p}_F^{\text{IS}}(\boldsymbol{Z}_{\boldsymbol{N^*=3}}^{(\boldsymbol{1})})$, $\widehat{p}_F^{\text{FORM}}(\boldsymbol{Z}_{\boldsymbol{N^*=3}}^{(\boldsymbol{1})})$ and $\gamma^*(\boldsymbol{Z}_{\boldsymbol{N^*=3}}^{(\boldsymbol{1})})$, as illustrated with the black arrows in the Figure. But when implementing the estimator of the failure probability by means of

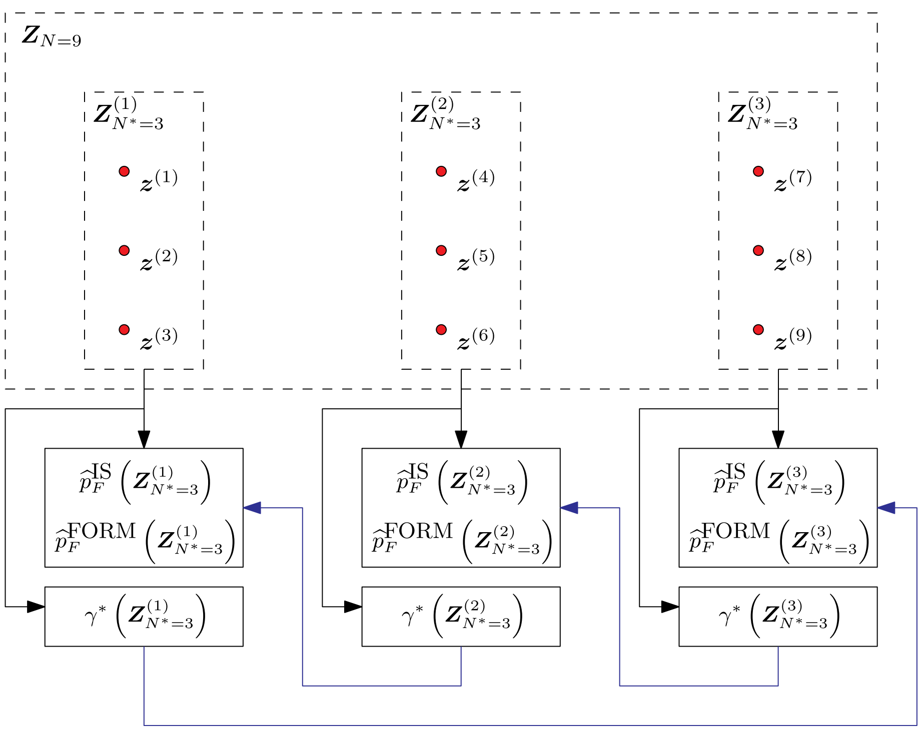

Figure 3. Schematic representation of Splitting technique.

CVS as shown in Equation (18), the estimators $\widehat{p}_F^{\mathrm{IS}}(\boldsymbol{Z}_{\boldsymbol{N^*=3}}^{(\boldsymbol{1})})$ and $\widehat{p}_F^{\mathrm{FORM}}(\boldsymbol{Z}_{\boldsymbol{N^*=3}}^{(\boldsymbol{1})})$ are controlled with the optimal control parameter coming from the second subset $\gamma^*(\boldsymbol{Z}_{\boldsymbol{N^*=3}}^{(\boldsymbol{2})})$, as illustrated with the blue arrow in Figure 3.

It is also possible to estimate the variance associated with the failure probability estimator in Equation (18), which is equal to, see e.g. Avramidis and Wilson (1993):

$$\begin{aligned}\widehat{\sigma}^2\left[\widehat{p}_F^{\mathrm{CVS}}\right] = \frac{1}{N_S^2}\sum_{k=1}^{N_S}\Big(&\widehat{\sigma}^2\left[\widehat{p}_F^{\mathrm{IS}}\left(\boldsymbol{Z}_{N^*}^{(k)}\right)\right] - 2\gamma^*\left(\boldsymbol{Z}_{N^*}^{\tau(k)}\right)\widehat{\delta}\left[\widehat{p}_F^{\mathrm{IS}}\left(\boldsymbol{Z}_{N^*}^{(k)}\right), \widehat{p}_F^{\mathrm{FORM}}\left(\boldsymbol{Z}_{N^*}^{(k)}\right)\right]\\ &+\left(\gamma^*\left(\boldsymbol{Z}_{N^*}^{\tau(k)}\right)\right)^2\widehat{\sigma}^2\left[\widehat{p}_F^{\mathrm{FORM}}\left(\boldsymbol{Z}_{N^*}^{(k)}\right)\right]\Big)\end{aligned} \tag{21}$$

where the expressions for evaluating the variance and covariance can be found in the Appendix.

As a summary of the above discussion, it is important to note that the application of the Splitting scheme within Control Variates does not demand additional evaluations of the performance function of the system being studied. In fact, the only additional step required for implementing the Splitting scheme is to keep track of the sample subsets and the associated estimators. Such step entails negligible numerical costs.

4.4. Estimation of the sensitivity of the failure probability using control variates with splitting

The expressions for calculating the sensitivity of the failure probability are quite similar to those involving the failure probability. Thus, Control Variates with Splitting can be applied directly to estimate the probability sensitivity. Before presenting such estimator, consider the following definition, which is actually the counterpart of the definition introduced in Equation (13).

$$\frac{\partial\widehat{p}_F^{\mathrm{FORM}}}{\partial\theta_{l,i}}(\boldsymbol{Z}_N) = \frac{1}{N}\sum_{j=1}^{N} I_{\boldsymbol{z}}^{\mathrm{FORM}}\left(\boldsymbol{z}^{(j)}\right)\eta_{z_i,\theta_{l,i}}\left(z_i^{(j)} \mid \boldsymbol{\theta}_i\right)\frac{\phi_n\left(\boldsymbol{z}^{(j)}\right)}{\phi_n\left(\boldsymbol{z}^{(j)} \mid \boldsymbol{z}^*\right)}, \tag{22}$$

Taking into account this definition, the expression for calculating probability sensitivity using Control Variates with Splitting is:

$$\begin{aligned}\frac{\partial\widehat{p}_F^{\mathrm{CVS}}}{\partial\theta_{l,i}} = \frac{1}{N_S}\sum_{k=1}^{N_S}\Bigg(&\frac{\partial\widehat{p}_F^{\mathrm{IS}}}{\partial\theta_{l,i}}\left(\boldsymbol{Z}_{N^*}^{(k)}\right) - \epsilon^*\left(\boldsymbol{Z}_{N^*}^{\tau(k)}\right)\frac{\partial\widehat{p}_F^{\mathrm{FORM}}}{\partial\theta_{l,i}}\left(\boldsymbol{Z}_{N^*}^{(k)}\right)\\ &+\epsilon^*\left(\boldsymbol{Z}_{N^*}^{\tau(k)}\right)\frac{\partial p_F^{\mathrm{FORM}}}{\partial\theta_{l,i}}\Bigg),\end{aligned} \tag{23}$$

where $\partial\widehat{p}_F^{\mathrm{IS}}/\partial\theta_{l,i}$ and $\partial\widehat{p}_F^{\mathrm{FORM}}/\partial\theta_{l,i}$ are the estimators in Equations (12) and (22) evaluated with the sample subset $\boldsymbol{Z}_{N^*}^{(k)}$; $\epsilon^*(\boldsymbol{Z}_{N^*}^{\tau(k)})$ is the optimal control parameter for sensitivity analysis associated with the sample subset $\boldsymbol{Z}_{N^*}^{\tau(k)}$, which is defined as:

$$\epsilon^*\left(\boldsymbol{Z}_{N^*}^{\tau(k)}\right) = \frac{\widehat{\delta}\left[\dfrac{\partial\widehat{p}_F^{\mathrm{IS}}}{\partial\theta_{l,i}}\left(\boldsymbol{Z}_N^{\tau(k)}\right), \dfrac{\partial\widehat{p}_F^{\mathrm{FORM}}}{\partial\theta_{l,i}}\left(\boldsymbol{Z}_N^{\tau(k)}\right)\right]}{\widehat{\sigma}^2\left[\dfrac{\partial\widehat{p}_F^{\mathrm{FORM}}}{\partial\theta_{l,i}}\left(\boldsymbol{Z}_N^{\tau(k)}\right)\right]}. \tag{24}$$

The estimator of the variance associated with the sensitivity in Equation (23) is:

$$\widehat{\sigma}^2\left[\frac{\partial \widehat{p}_F^{CVS}}{\partial \theta_{l,i}}\right] = \frac{1}{N_S^2}\sum_{k=1}^{N_S}\Bigg(\widehat{\sigma}^2\left[\frac{\partial \widehat{p}_F^{IS}}{\partial \theta_{l,i}}\left(\boldsymbol{Z}_{N^*}^{(k)}\right)\right] - 2\epsilon^*\left(\boldsymbol{Z}_{N^*}^{\tau(k)}\right)\widehat{\delta}\left[\frac{\partial \widehat{p}_F^{IS}}{\partial \theta_{l,i}}\left(\boldsymbol{Z}_{N^*}^{(k)}\right), \frac{\partial \widehat{p}_F^{FORM}}{\partial \theta_{l,i}}\left(\boldsymbol{Z}_{N^*}^{(k)}\right)\right] + \left(\epsilon^*\left(\boldsymbol{Z}_{N^*}^{\tau(k)}\right)\right)^2\widehat{\sigma}^2\left[\frac{\partial \widehat{p}_F^{FORM}}{\partial \theta_{l,i}}\left(\boldsymbol{Z}_{N^*}^{(k)}\right)\right]\Bigg) \quad (25)$$

Expressions for evaluating the variance and covariance appearing in the last equation can be found in the Appendix.

4.5. Summary

The procedure for estimating probability and its sensitivity by synthesising the results of FORM and IS with Control Variates with Splitting can be summarised in the following three main steps.

(1) *First-Order Reliability Method*. Conduct an ordinary analysis with FORM. This implies identifying the design point with any appropriate algorithm, as well as calculating the failure probability with Equation (7) and its sensitivity with Equation (8).
(2) *Importance Sampling Using Design Points*. Conduct an ordinary analysis with IS as described in Section 3.3. This implies generating samples of the uncertain variables according to the prescribed importance sampling density function, as well as calculating the failure probability with Equation (11) and its sensitivity with Equation (12).
(3) *Control Variates with Splitting*. Post-process the results of the previous two steps. Set the number of subsets $N_S = 3$. To estimate the failure probability and its variance, apply Equations (18) and (21), respectively. To estimate the sensitivity of the failure probability and its variance, apply Equations (23) and (25), respectively.

It is emphasised that evaluations of the performance function take place only in the first two steps. The third step of Control Variates with Splitting can be interpreted merely as a post-processing step, aiming at leveraging on the probability estimate and its sensitivity obtained with FORM and IS.

5. Examples

5.1. Example 1: analytic performance function

This example involves the following explicit performance function:

$$g_{\boldsymbol{x}}(\boldsymbol{x}) = 3 - x_1 + \kappa x_2^2 \quad (26)$$

where x_1 and x_2 are uncertain input variables characterised through standard normal random variables; and κ is a real value that adjusts the degree of non-linearity of the associated limit state function. The objective is calculating both the failure probability and its sensitivity with respect to the mean value of the first random variable μ_1.

Table 1. Estimates for failure probability and its sensitivity considering $\kappa = 0.05$ – Example 1.

Approach	p_F	CoV(p_F)	$\partial p_F/\partial\mu_1$	CoV($\partial p_F/\partial\mu_1$)	N
Monte Carlo	1.2×10^{-3}	0.9%	3.9×10^{-3}	0.9%	10^7
FORM	1.3×10^{-3}	–	4.4×10^{-3}	–	2
IS	9.9×10^{-4}	21.3%	3.3×10^{-3}	20%	2+90
CVS	1.2×10^{-3}	8.2%	4.0×10^{-3}	7.7%	2+90

To solve this problem, κ is first chosen as 0.05. The results obtained by means of different methods are reported in Table 1. This table summarises the different approaches used to solve the reliability problem, the estimates for the failure probability (p_F) and the sought sensitivity ($\partial p_F/\partial\mu_1$), the corresponding coefficients of variation (CoV(·)) and the number of performance function evaluations (N). Please recall that the coefficient of variation is equal to the standard deviation of an estimate divided by its expected value.

The results obtained with Monte Carlo simulation as reported in Table 1 are regarded as a reference. It is noted that FORM provides good estimates for the failure probability (relative error of about 8%) and its sensitivity (relative error of about 15%). This was expected as the limit state function is slightly non-linear. Moreover, only two evaluations of the performance function are required to determine the design point. This is because the search of the design point is performed with gradient information, which is derived analytically due to the simplicity of the performance function. The downside of FORM is that its accuracy is unknown. IS is implemented using 90 simulations and provides reasonable estimates of the failure probability and its sensitivity, although the coefficients of variation are relatively high, revealing lack of precision. CVS makes optimal use of the information provided by FORM and IS, as it provides estimates which are accurate (close to the reference result) and precise (small coefficient of variation).

In a next step, the estimation of probability and its sensitivity is repeated, but this time considering $\kappa = 0.3$. The results obtained are shown in Table 2. Overall, the tendencies observed in Table 1 are repeated. However, there are two issues to be noted. First, the results of FORM do not vary between both cases. This is a consequence of the fact that the design point remains the same for the two values of κ considered. Furthermore, this is reflected in the increase in the relative error of the estimates, which is about 68% for the failure probability and 69% for the probability sensitivity when $\kappa = 0.3$. Second, while CVS offers improvement over IS, this improvement is less pronounced than in the previous case. This occurs because the FORM hypothesis deviates significantly from the true limit state surface when $\kappa = 0.3$.

As an additional remark, please note that all results reported in Tables 1 and 2 have been produced with a single simulation run (that is, IS and CVS are run only once with a sample size $N = 90$). This emphasises the fact that the implementation of CVS as reported in this work produces estimates of the sought quantities (failure probability

Table 2. Estimates for failure probability and its sensitivity considering $\kappa = 0.3$ – Example 1.

Approach	p_F	CoV(p_F)	$\partial p_F/\partial\mu_1$	CoV($\partial p_F/\partial\mu_1$)	N
Monte Carlo	7.7×10^{-4}	1.1%	2.6×10^{-3}	1.1%	10^7
FORM	1.3×10^{-3}	–	4.4×10^{-3}	–	2
IS	6.7×10^{-4}	23.8%	2.3×10^{-3}	22.5%	2+90
CVS	8.4×10^{-4}	18.1%	2.8×10^{-3}	17.1%	2+90

and its sensitivity) as well as estimates of their coefficient of variation with a single simulation run.

5.2. Example 2: confined seepage below a dam

This example is taken from Valdebenito et al. (2018) and involves analyzing the seepage below an impermeable dam that rests over two soil layers whose permeability is uncertain. The problem is depicted schematically in Figure 4. The dam rests over two soil layers, where the top layer possesses a depth 5 [m] while the bottom layer's depth is 15 [m]. The vertical and horizontal permeabilities of each soil layer are modelled as lognormal random variables with coefficient of variation 30%. The mean values of the horizontal and vertical permeabilities are $(\mu_{k_{xx,1}}, \mu_{k_{yy,1}}) = (5, 2) \times 10^{-7}$ [m/s] for the top layer and $(\mu_{k_{xx,2}}, \mu_{k_{yy,2}}) = (5, 2) \times 10^{-6}$ [m/s] for the bottom layer.

The upstream side withstands a water column of 10 [m] height while there is no water column at downstream side. All boundaries of the model are assumed as impermeable except for the upstream and downstream sides. To calculate the seepage flow, a finite element model with 1628 quadratic triangular elements is considered.

The objective of this example is twofold: calculation the probability that the seepage flow below the dam exceeds the threshold of 18 [L/h/m] and the sensitivity of that probability with respect to the mean permeability of the top soil layer $\mu_{k_{xx,1}}$. Estimates for both quantities are produced using FORM, IS and CVS. On one side, the implementation of FORM demanded a total of 5 system analyses for calculating the design point. Please note that gradients of the performance function were calculated as a by-product of those system analyses using expressions available for linear systems, see e.g. Haftka and Gürdal (1992). On the other side, IS is implemented considering 150 samples. A total of 300 independent runs are considered to assess the robustness of the estimators. The results obtained are reported in Figure 5 in terms of a normalised histograms for IS and CVS and a vertical line for FORM (as the latter provides crisp estimators). The normalisation is such that the area below the histograms is equal to 1, so that they resemble the probability density function (PDF) associated with the different estimators. The results show that for both the probability and its sensitivity, the results obtained with CVS present less variability than those associated with IS. This highlights the potential of CVS for producing more accurate results than those associated with IS. In addition, it is observed there are discrepancies between the results of FORM and IS/CVS. This was expected due to the linearisation hypothesis behind FORM.

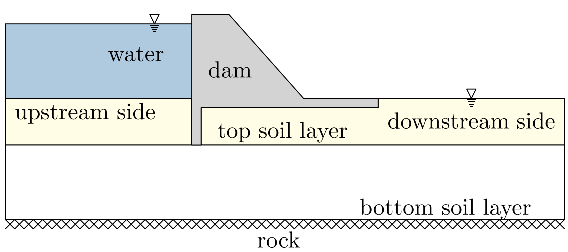

Figure 4. Schematic representation of impermeable dam resting over two permeable soil layers.

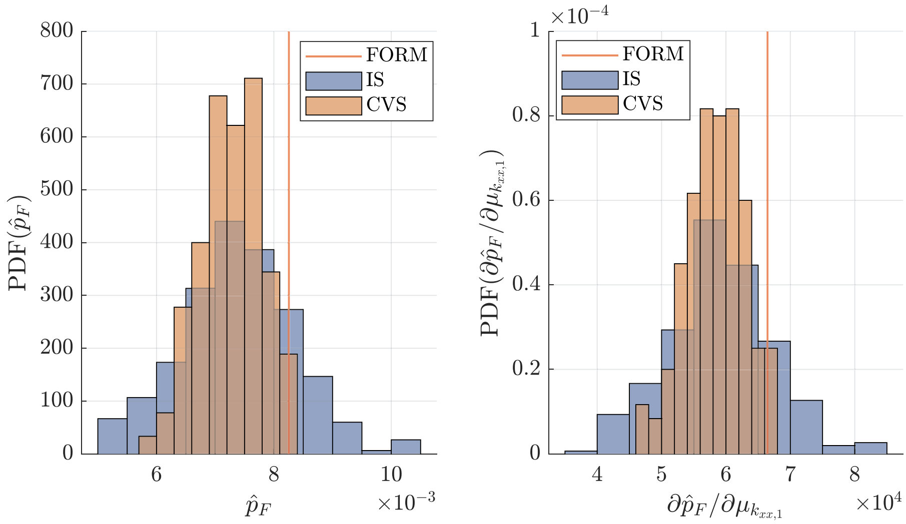

Figure 5. Probability density functions associated with estimators of failure probability and its sensitivity.

6. Conclusions

This work has discussed the aggregation of the results produced with FORM and Importance Sampling for estimating failure probabilities and their sensitivity. Such aggregation is carried out resorting to the framework provided by Control Variates with Splitting. The results obtained show that such a framework may indeed lead to improved estimates, which possess both more precision and more accuracy than that associated with the estimators produced with a reliability method alone. However, the level of improvement attained is directly connected with the quality of the FORM approximation. Indeed, for those cases where the FORM approximation provides a reasonable approximation of the limit state surface, it is expected that Control Variates with Splitting will offer significant synergies. Moreover, it is noted that the framework provided by Control Variates with Splitting does not demand any additional evaluations of the performance function. Hence, the scheme developed in this contribution can be regarded as a post-processing step of Importance Sampling using design points.

There are several paths for future development of Control Variates with Splitting. One possibility would be exploring its application with alternative approximate reliability methods such as the Second-Order Reliability Method (SORM). Another path for development consists of investigating non-linear versions of Control Variates, which may be helpful for addressing more general types of reliability problems. Yet another path for development is extending the framework implemented in this work for calculating the sensitivity of the failure probability with respect to deterministic parameters that affect the performance function. Usually, calculating such sensitivity is more involved than calculating sensitivity with respect to distribution parameters (Rubinstein and Kroese 2007; Torii and Novotny 2021; Wu 1994).

Disclosure statement

No potential conflict of interest was reported by the author(s).

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Appendix. Expressions for Calculating Variance and Covariance of Probability Estimators

Estimates of the variances associated with $\widehat{p}_F^{\text{IS}}(\boldsymbol{Z}_N)$ (see Equation (11)) and $\partial\widehat{p}_F^{\text{IS}}/\partial\theta_{l,i}(\boldsymbol{Z}_N)$ (see Equation (12)) are given by the following expressions.

$$\widehat{\sigma}^2\left[\widehat{p}_F^{\text{IS}}(\boldsymbol{Z}_N)\right] = \frac{1}{N(N-1)}\sum_{j=1}^{N}\left(I_{\boldsymbol{z}}\left(\boldsymbol{z}^{(j)}\right)\frac{\phi_n\left(\boldsymbol{z}^{(j)}\right)}{\phi_n\left(\boldsymbol{z}^{(j)}\mid\boldsymbol{z}^*\right)} - \widehat{p}_F^{\text{IS}}(\boldsymbol{Z}_N)\right)^2, \tag{A1}$$

$$\widehat{\sigma}^2\left[\frac{\partial\widehat{p}_F^{\text{IS}}}{\partial\theta_{l,i}}(\boldsymbol{Z}_N)\right] = \frac{1}{N(N-1)}\sum_{j=1}^{N}\left(I_{\boldsymbol{z}}\left(\boldsymbol{z}^{(j)}\right)\eta_{z_i,\theta_{l,i}}\left(z_i^{(j)}\mid\boldsymbol{\theta}_i\right)\frac{\phi_n\left(\boldsymbol{z}^{(j)}\right)}{\phi_n\left(\boldsymbol{z}^{(j)}\mid\boldsymbol{z}^*\right)} - \frac{\partial\widehat{p}_F^{\text{IS}}}{\partial\theta_{l,i}}(\boldsymbol{Z}_N)\right)^2. \tag{A2}$$

Estimates of the variances associated with $\widehat{p}_F^{\text{FORM}}(\boldsymbol{Z}_N)$ (see Equation (13)) and $\partial\widehat{p}_F^{\text{FORM}}/\partial\theta_{l,i}(\boldsymbol{Z}_N)$ (see Equation (22)) are given by the following expressions.

$$\widehat{\sigma}^2\left[\widehat{p}_F^{\text{FORM}}(\boldsymbol{Z}_N)\right] = \frac{1}{N(N-1)}\sum_{j=1}^{N}\left(I_{\boldsymbol{z}}^{\text{FORM}}\left(\boldsymbol{z}^{(j)}\right)\frac{\phi_n\left(\boldsymbol{z}^{(j)}\right)}{\phi_n\left(\boldsymbol{z}^{(j)}\mid\boldsymbol{z}^*\right)} - \widehat{p}_F^{\text{FORM}}(\boldsymbol{Z}_N)\right)^2, \tag{A3}$$

$$\widehat{\sigma}^2\left[\frac{\partial\widehat{p}_F^{\text{FORM}}}{\partial\theta_{l,i}}(\boldsymbol{Z}_N)\right] = \frac{1}{N(N-1)}\sum_{j=1}^{N}\left(I_{\boldsymbol{z}}^{\text{FORM}}\left(\boldsymbol{z}^{(j)}\right)\eta_{z_i,\theta_{l,i}}\left(z_i^{(j)}\mid\boldsymbol{\theta}_i\right)\frac{\phi_n\left(\boldsymbol{z}^{(j)}\right)}{\phi_n\left(\boldsymbol{z}^{(j)}\mid\boldsymbol{z}^*\right)} - \frac{\partial\widehat{p}_F^{\text{FORM}}}{\partial\theta_{l,i}}(\boldsymbol{Z}_N)\right)^2. \tag{A4}$$

An estimate of the covariance between $\widehat{p}_F^{\text{IS}}(\boldsymbol{Z}_N)$ and $\widehat{p}_F^{\text{FORM}}(\boldsymbol{Z}_N)$ is the following.

$$\widehat{\delta}\left[\widehat{p}_F^{\text{IS}}(\boldsymbol{Z}_N),\ \widehat{p}_F^{\text{FORM}}(\boldsymbol{Z}_N)\right] = \frac{1}{N(N-1)}\sum_{j=1}^{N}\left(I_{\boldsymbol{z}}\left(\boldsymbol{z}^{(j)}\right)\frac{\phi_n\left(\boldsymbol{z}^{(j)}\right)}{\phi_n\left(\boldsymbol{z}^{(j)}\mid\boldsymbol{z}^*\right)} - \widehat{p}_F^{\text{IS}}(\boldsymbol{Z}_N)\right)\left(I_{\boldsymbol{z}}^{\text{FORM}}\left(\boldsymbol{z}^{(j)}\right)\frac{\phi_n\left(\boldsymbol{z}^{(j)}\right)}{\phi_n\left(\boldsymbol{z}^{(j)}\mid\boldsymbol{z}^*\right)} - \widehat{p}_F^{\text{FORM}}(\boldsymbol{Z}_N)\right) \tag{A5}$$

An estimate of the covariance between $\partial\widehat{p}_F^{\text{IS}}/\partial\theta_{l,i}(\boldsymbol{Z}_N)$ and $\partial\widehat{p}_F^{\text{FORM}}/\partial\theta_{l,i}(\boldsymbol{Z}_N)$ is the following.

$$\widehat{\delta}\left[\frac{\partial\widehat{p}_F^{\text{IS}}}{\partial\theta_{l,i}}(\boldsymbol{Z}_N),\ \frac{\partial\widehat{p}_F^{\text{FORM}}}{\partial\theta_{l,i}}(\boldsymbol{Z}_N)\right] = \frac{1}{N(N-1)}\sum_{j=1}^{N}\left(I_{\boldsymbol{z}}\left(\boldsymbol{z}^{(j)}\right)\eta_{z_i,\theta_{l,i}}\left(z_i^{(j)}\mid\boldsymbol{\theta}_i\right)\frac{\phi_n\left(\boldsymbol{z}^{(j)}\right)}{\phi_n\left(\boldsymbol{z}^{(j)}\mid\boldsymbol{z}^*\right)} - \frac{\partial\widehat{p}_F^{\text{IS}}}{\partial\theta_{l,i}}(\boldsymbol{Z}_N)\right)\left(I_{\boldsymbol{z}}^{\text{FORM}}\left(\boldsymbol{z}^{(j)}\right)\eta_{z_i,\theta_{l,i}}\left(z_i^{(j)}\mid\boldsymbol{\theta}_i\right)\frac{\phi_n\left(\boldsymbol{z}^{(j)}\right)}{\phi_n\left(\boldsymbol{z}^{(j)}\mid\boldsymbol{z}^*\right)} - \frac{\partial\widehat{p}_F^{\text{FORM}}}{\partial\theta_{l,i}}(\boldsymbol{Z}_N)\right) \tag{A6}$$