Augmented First-Order Reliability Method for Estimating Fuzzy Failure Probabilities

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Abstract

This paper presents an approach for estimating the fuzzy failure probability associated with reliability problems where uncertainty is characterized through random variables whose distribution parameters are described as fuzzy variables. The main contribution of this work is addressing such problem with the First-Order Reliability Method, with some minor modifications. The epistemic uncertainty is addressed by resorting to an augmented reliability problem. In this way, a single reliability analysis suffices for estimating the membership function associated with the fuzzy failure probability. Numerical results suggest that the proposed approach is most useful for providing an estimate of the membership function associated with the failure probability with reduced numerical costs.

Keywords: Fuzzy failure probability, Augmented reliability problem, First-Order Reliability Method

Highlights:

- Focus on calculation of fuzzy failure probabilities.
- Augmented reliability problem allows accounting for epistemic uncertainty.
- First-Order Reliability Method is applied to solve augmented problem.
- Single reliability analysis suffices for approximating membership function.

1. Introduction

Probability theory has become a widespread means for characterizing uncertainty associated with practical engineering systems [1]. Following this framework, the uncertainty associated with parameters that affect the performance of a system is described in terms of probability distributions. Thus, the level of safety of a system can be quantified in terms of a failure probability, that

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measures the chances that the performance of the system undergoes an undesirable behavior [2]. For cases of practical interest, the calculation of this failure probability is a challenging task and therefore, several specialized methods have been devised for its calculation, involving approximation concepts (see, e.g. [3, 4]), surrogate modeling (see, e.g. [5, 6]), probability density evolution methods (see, e.g. [7–9]) and simulation approaches (see, e.g. [10, 11]).

The above discussion assumes that uncertainty associated with a problem is aleatory, that is, due to randomness. However, it has been acknowledged that uncertainties may be of the epistemic type [12] whenever they originate from issues such as lack of knowledge, imprecision, etc. Such type of uncertainty may be described resorting to models such as intervals or fuzzy variables, as discussed in e.g. [13]. More often than not, one may be confronted with both types of uncertainties in a practical situation. Under such circumstances, hybrid uncertainty models become most useful (see, e.g. [14]). In particular, fuzzy probability (see, e.g.[15]) constitutes a straightforward yet powerful framework for modeling and capturing the effects of aleatory and epistemic uncertainty. In essence, fuzzy probability consists of modeling the uncertainty associated with an input parameter of a problem with a probability density function whose distribution parameters (for example, mean and/or standard deviation) are characterized as fuzzy variables (see also [16] for a review on recent computational methods). Under such assumption, the failure probability associated with a reliability problem is no longer a crisp value but instead, it becomes a fuzzy variable with its own membership function [17].

The practical deployment of fuzzy probabilities is usually a challenging task. This stems out of the necessity of quantifying the effects of both aleatory and epistemic uncertainty, but without mixing them. The most direct means for solving problems of fuzzy probabilities is implementing a nested approach, which involves exploring the space of epistemic parameters and performing classical reliability analysis (the latter for fixed values of the epistemic parameters). However, this type of approach can become extremely demanding from a numerical viewpoint, as it requires repeated deterministic system analyses for different realizations of the epistemic distribution parameters and aleatory input parameters. In view of this challenge, several different specialized approaches have been developed to cope with probability estimation under aleatory and epistemic uncertainties. These approaches encompass, for example, optimization techniques [18, 19], sampling strategies [20–25], approximation concepts [26], meta-models [27, 28], decoupling strategies [29], interval analysis and simulation [30], etc. All of these works attest the enormous progress that has been achieved in this area. Nonetheless, there are two issues where additional research is required. First, it is not obvious how to bring the calculation of fuzzy probabilities as close as possible to existing reliability methods. Indeed, addressing such an issue may expedite the application of fuzzy probabilities by practitioners. Second, there exists still a need to decrease numerical efforts, such that estimating fuzzy probabilities is (ideally) not substantially more demanding than performing a classical reliability in mind. With these two issues in mind, this contribution attempts to close this gap by proposing an approach for reliability analysis considering fuzzy probabilities. More specifically, the objective is approximating the membership function associated with the failure probability for a problem where the input parameters of a numerical model are characterized through random variables whose distribution parameters (e.g. mean and/or standard deviation) are described through fuzzy variables. Epistemic uncertainties are reckoned by means of an augmented reliability problem [25, 31], which is solved using the First-Order Reliability Method (FORM, see e.g. [32]). In this way, it is possible to roughly approximate the membership function associated with the failure probability at strongly reduced numerical costs, as only a single reliability analysis is required. The expected range of application of the proposed approach is similar to that of FORM for classical reliability analysis (see, e.g. [33]). This is due to the fact that the fuzzy probability problem is projected into the standard normal space and in addition, the limit state surface is linearized about the so-called design point. Furthermore, the practical implementation of the proposed approach is quite straightforward, as it demands minor modifications with respect to a classical reliability analysis conducted with FORM. In this sense, it is expected that this contribution may be useful for practitioners who want to perform a sensitivity analysis with respect to sources of epistemic uncertainties by means of well-known, classical tools for reliability analysis.

It is important to note that the application of FORM for reliability analysis under epistemic uncertainty has already been explored in the past. For example, in [18, 34–36], the calculation of a bounding value for the probability entails solving an optimization problem involving aleatory and epistemic parameters. In [37], bounding values for the failure probability are determined by locating a representative design point associated with the FORM approximation. In contrast, the approach proposed in this work allows obtaining an explicit approximation of the failure probability as a function of the epistemic parameters once the augmented reliability problem is solved. Therefore, bounding values for the failure probability can be determined in closed form by using the aforementioned explicit approximation.

This paper is organized as follows. Section 2 formulates the reliability problem involving fuzzy probabilities. Section 3 discusses the formulation of an augmented reliability problem which is solved with FORM and that allows approximating the failure probability as an explicit function of the epistemic parameters. The application of the approach developed in Section 3 is illustrated by means of a numerical example in Section 4. Finally, the paper closes with discussions and conclusions in Section 5.

2. Formulation of the problem

2.1. Failure Probability as a Function of the Distribution Parameters

Consider an engineering system which is represented using an appropriate numerical model by means of, e.g. the finite element method [38]. The behavior of this system is characterized in terms of the so-called performance function g, which is dependent on a set of input parameters collected in vector $\boldsymbol{x}$ of dimension $n_x \times 1$, where $\boldsymbol{x} \in \Omega_X$. Whenever a particular value of these input parameters causes an undesirable behavior of the system, the performance function assumes a value equal or smaller than zero, that is, $g(\boldsymbol{x}) \leq 0$. It is assumed that the input parameters are not known precisely and their (aleatory) uncertainty is characterized by means of a random variable vector $\boldsymbol{X}$ with joint probability distribution $f_{\boldsymbol{X}}(\boldsymbol{x}|\boldsymbol{\theta})$, where $\boldsymbol{\theta}$ denotes a vector of distribution parameters of dimension $n_\theta \times 1$, which contains values such as mean, standard deviation, etc. Considering the previous assumptions, the probability of failure p_F associated with the system given $\boldsymbol{\theta}$ is expressed as (see, e.g. [39, 40]):

$$p_F(\boldsymbol{\theta}) = \int_{g(\boldsymbol{x}) \leq 0} f_{\boldsymbol{X}}(\boldsymbol{x}|\boldsymbol{\theta}) d\boldsymbol{x}. \tag{1}$$

As noted from eq. (1), the failure probability depends on the value assumed by the distribution parameters $\boldsymbol{\theta}$. This makes sense from a physical viewpoint. For example, it is expected that changes in the expected value of a physical parameter can significantly affect the numerical value of the failure probability.

2.2. Fuzzy Distribution Parameters

In practical situations, establishing crisp values for these distribution parameters $\boldsymbol{\theta}$ may be a challenging task due to issues such as lack of knowledge, the scarceness and/or imprecision of measurements, etc. Such type of uncertainty corresponds to the epistemic type and can be characterized, for example, by means of fuzzy variables (see, e.g. [17]). Thus, each distribution parameter is described in terms of a fuzzy set $\tilde{\theta}_l$:

$$\tilde{\theta}_l = \left\{ \left(\theta_l, \mu_{\tilde{\theta}_l}(\theta_l)\right) : (\theta_l \in \Theta_l) \wedge \left(\mu_{\tilde{\theta}_l}(\theta_l) \in [0,1]\right) \right\}, \ l = 1, \ldots, n_\theta, \tag{2}$$

where θ_l denotes the value of the l-th distribution parameter; Θ_l denotes the set of possible values that θ_l may assume; and $\mu_{\tilde{\theta}_l}(\theta_l)$ is the membership function. The fuzzy set $\tilde{\theta}_l$ in eq. (2) assigns a *membership* to each value contained in Θ_l, where *membership* is understood as the degree with which θ_l belongs to $\tilde{\theta}_l$. In this contribution, it is assumed that fuzzy variables possess a triangular membership function (see, e.g. [17]). In such case, the membership is characterized by its lower bound $\underline{\theta}_l$ and upper bound $\overline{\theta}_l$ for which $\mu_{\tilde{\theta}_l}(\underline{\theta}_l) = \mu_{\tilde{\theta}_l}(\overline{\theta}_l) = 0$; and the so-called mean value $\theta_{l,\mu}$ for which $\mu_{\tilde{\theta}_l}(\theta_{l,\mu}) = 1$; these data are written in compact form as $\langle \underline{\theta}_l, \theta_{l,\mu}, \overline{\theta}_l \rangle$. Figure 1 contains a schematic representation of a triangular membership function.

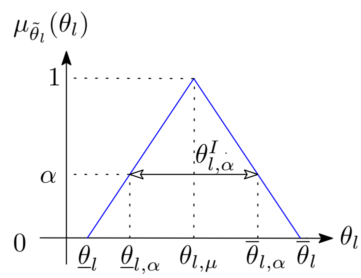

Figure 1: Fuzzy triangular membership function.

The type of membership considered in this work correspond to a convex one. This implies that for a particular membership level $\alpha \in (0,1]$, one may extract an interval $\theta^I_{l,\alpha}$ whose lower and upper bounds are $\underline{\theta}_{l,\alpha}$ and $\overline{\theta}_{l,\alpha}$, respectively. This interval $\theta^I_{l,\alpha}$ is depicted schematically in Figure 1. Hence, a fuzzy set with a convex membership function can be actually interpreted as a collection of intervals indexed by the membership level α [41]. Note that there are several types of membership functions which are convex, such as triangular or trapezoidal ones. However, in this work, the focus is exclusively on triangular membership functions. The reason is that such class of membership function is most useful for cases where limited information concerning the fuzzy variables is available.

2.3. Fuzzy Probability

Problems where uncertainty on the input parameters $\boldsymbol{x}$ of a model is characterized in terms of a probability density function $f_{\boldsymbol{X}}(\boldsymbol{x}|\boldsymbol{\theta})$ while uncertainty on the distribution parameters $\boldsymbol{\theta}$ is characterized in terms of fuzzy sets $\tilde{\boldsymbol{\theta}}$ fall in the category of fuzzy probabilities, see e.g. [15]. Indeed, this class of problems can be interpreted as a collection of probability models which are indexed by the distribution parameters and is able to capture aleatory and epistemic uncertainty, without mixing them. Under such model for uncertainty characterization, the failure probability as cast in eq. (1) becomes a fuzzy variable as well and as such, it possesses its own membership function $\mu_{\tilde{p}_F}(p_F)$. Determining the membership function $\mu_{\tilde{p}_F}(p_F)$ is of much relevance, as it provides a type of sensitivity measure which reveals how sensitive the failure probability is with respect to the epistemic uncertainty associated the distribution parameters.

2.4. α-Level Optimization

Different methods have been developed for coping with fuzzy sets and calculating the membership function of quantities of interest, see e.g. [42]. In this work, the membership function is determined using the so-called α-level optimization [43]. The basis of α-level optimization is focusing on a particular membership level $\alpha \in (0, 1]$. As already discussed in Section 2.2, for a given membership level α and in view of convexity, the uncertainty associated with the l-th distribution parameter can be interpreted as an interval $\theta^I_{l,\alpha}$. Therefore, for that particular membership level α, it is possible to assess the interval associated with the failure probability $p^I_{F,\alpha}$, whose lower $\underline{p}_{F,\alpha}$ and upper bounds $\overline{p}_{F,\alpha}$ are equal to:

$$\underline{p}_{F,\alpha} = \min_{\boldsymbol{\theta}\in\boldsymbol{\theta}^I_\alpha} \left(p_F(\boldsymbol{\theta})\right) \tag{3}$$

$$\overline{p}_{F,\alpha} = \max_{\boldsymbol{\theta}\in\boldsymbol{\theta}^I_\alpha} \left(p_F(\boldsymbol{\theta})\right) \tag{4}$$

where $\boldsymbol{\theta}^I_\alpha$ denotes the hyper-rectangle defined by the individual intervals $\theta^I_{l,\alpha},\ l = 1, \ldots n_\theta$. The pair of eqs. (3) and (4) actually corresponds to an interval analysis carried out at the given α-level, where the bounds are determined by means of optimization. This interval analysis can be repeated for different values of the membership level in order to produce a collection of intervals for the failure probability. This collection provides a discrete approximation of the membership function associated with the failure probability. This concept is depicted schematically in Figure 1, where for simplicity, only two α-levels α_1 and α_2 have been considered.

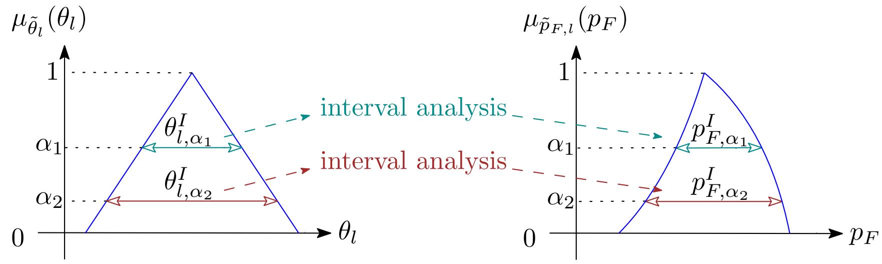

Figure 2: α-Level optimization.

The α-Level optimization process as described in eqs. (3) and (4) and Figure 2 can be interpreted as a triple-loop procedure. Indeed, in the outer loop, different membership values α are swept in order to produce a discrete approximation of the membership function associated with the failure probability. In the middle loop, different realizations of the distribution parameters $\boldsymbol{\theta}$ are explored in order to determine the extrema of the failure probability such that $\boldsymbol{\theta} \in \boldsymbol{\theta}^I_\alpha$. In the inner loop and given a specific realization of the distribution parameters, a classical reliability analysis is performed. For most problems of practical interest, even a classical reliability analysis with fixed distribution parameters is quite demanding from a numerical viewpoint. Therefore, it is expected that the numerical costs associated with the calculation of the membership function of the failure probability may become extremely high. In view of this challenge, this contribution proposes an approach for significantly decreasing numerical efforts, as described in the sequence.

3. Augmented First-Order Reliability Method

3.1. General Remarks

This Section presents an approach termed as augmented First-order Reliability Method (aFORM) that is most useful to approximate the membership function associated with the failure probability. This approach is based on the reformulation of the imprecise probabilistic problem into an augmented reliability problem that is solved by means of FORM. The concept of the augmented reliability problem is first discussed in Section 3.2. Then, Section 3.3 discusses how to project the augmented reliability problem into the standard normal space. Section 3.4 discusses the solution of the augmented reliability problem in the standard normal space by means of FORM and also proposes an explicit approximation of the failure probability with respect to the distribution parameters. Finally, Section 3.5 discusses some aspects for the practical implementation of the proposed approach.

3.2. Augmented Reliability Problem

A possible means to speed up the calculation of the target membership function would be approximating the failure probability as an explicit function of the distribution parameters. With such an approximation, the solution of eqs. (3) and (4) would entail an almost negligible effort. Naturally, the major challenge for implementing such a strategy lies precisely in constructing the aforementioned explicit approximation. A possible means for achieving such goal is resorting to an augmented reliability problem (see, e.g. [31, 40, 44]), that consists of associating auxiliary probability distributions to each of the distribution parameters. The latter may seem contradictory, as the uncertainty associated with distribution parameters has already been characterized using fuzzy sets. However, the association of auxiliary probability distributions should be regarded as an artifact that is useful for the sole purpose of constructing the sought approximation of the failure probability [31]. While in principle there are several possible auxiliary probability distributions which could be associated with the distribution parameters, usually a uniform distribution is the simplest choice. Thus, the auxiliary probability density function $f_{\Theta_l}(\theta_l)$ associated with θ_l is:

$$f_{\Theta_l}(\theta_l) = \begin{cases} \frac{1}{\overline{\theta}_{l,D} - \underline{\theta}_{l,D}} & \theta_l \in \left[\underline{\theta}_{l,D}, \overline{\theta}_{l,D}\right] \\ 0 & \text{otherwise} \end{cases}, \; l = 1, \ldots, n_\theta, \tag{5}$$

where $\underline{\theta}_{l,D}$ and $\overline{\theta}_{l,D}$ denote the lower and upper bounds for the uniform distribution. The precise criterion for selecting these bounds is discussed in detail in Section 3.5. However, it should be noted that $\underline{\theta}_{l,D} \leq \underline{\theta}_l$ and $\overline{\theta}_{l,D} \geq \overline{\theta}_l$. That is, the support of the uniform distribution associated with θ_l should be equal or larger than the support associated with the triangular fuzzy variable $\tilde{\theta}_l$.

Once auxiliary probability distributions have been associated with each distribution parameter, it is possible to define the so-called augmented failure probability p_F^A (see, e.g. [40]):

$$p_F^A = \int_{g(\boldsymbol{x}) \leq 0} f_{\boldsymbol{X}}(\boldsymbol{x}|\boldsymbol{\theta}) f_{\boldsymbol{\Theta}}(\boldsymbol{\theta}) d\boldsymbol{x} d\boldsymbol{\theta}, \tag{6}$$

where $f_{\boldsymbol{\Theta}}(\boldsymbol{\theta})$ represents the product of the individual auxiliary probability density functions $f_{\Theta_l}(\theta_l)$, $l = 1, \ldots, n_\theta$. Note that the aim is not to calculate the augmented failure probability p_F^A. Instead, the objective is to construct an approximation of eq. (1) which is an explicit function of $\boldsymbol{\theta}$ by considering the augmented reliability problem, as discussed in detail below.

3.3. Transformation to Standard Normal Space

A convenient means for solving the augmented failure probability integral in eq. (6) is projecting it into the standard normal space. Under the simplifying assumption that all random variables contained in vector $\boldsymbol{X}$ are independent between them, the expressions for transforming the random variables involved in the augmented reliability problem into the standard normal space are:

$$z_{P,i} = \Phi^{-1}\left(F_{X_i}\left(x_i|\boldsymbol{\theta}^{(i)}\right)\right),\ i = 1,\ldots,n_x \tag{7}$$

$$z_{A,l} = \Phi^{-1}\left(F_{\Theta_l}\left(\theta_l\right)\right),\ l = 1,\ldots,n_\theta \tag{8}$$

where x_i denotes the i-th input parameter and is a realization of the random variable X_i with distribution parameters $\boldsymbol{\theta}^{(i)}$ and cumulative distribution function $F_{X_i}(\cdot)$; $F_{\Theta_l}(\cdot)$ denotes the uniform cumulative density function associated with the l-th distribution parameter θ_l; $\Phi(\cdot)^{-1}$ denotes the inverse of the standard normal cumulative distribution function; $z_{P,i}$ denotes the realization in the standard normal space associated with the pair $\left(x_i, \boldsymbol{\theta}^{(i)}\right)$; and $z_{A,l}$ denotes the realization in the standard normal space associated with θ_l. Appendix A provides explicit expressions for eqs. (7) and (8) for the specific case of normal and lognormal random variables.

The set of transformations given by eqs. (7) and (8) can be expressed in compact form as:

$$\boldsymbol{z} = \begin{bmatrix} \boldsymbol{z}_P \\ \boldsymbol{z}_A \end{bmatrix} = \begin{bmatrix} \boldsymbol{T}_{XP}(\boldsymbol{x}|\boldsymbol{\theta}) \\ \boldsymbol{T}_{\Theta A}(\boldsymbol{\theta}) \end{bmatrix} \tag{9}$$

where $\boldsymbol{z}_P = [z_{P,1},\ldots,z_{P,n_x}]^T$ and $\boldsymbol{z}_A = [z_{A,1},\ldots,z_{A,n_\theta}]^T$; and where $\boldsymbol{T}_{XP}(\boldsymbol{x}|\boldsymbol{\theta}) : \Omega_X \mapsto \mathbb{R}^{n_x}$ and $\boldsymbol{T}_{\Theta A}(\boldsymbol{\theta}) : [\underline{\theta}_{1,D}, \overline{\theta}_{1,D}] \times \ldots \times [\underline{\theta}_{n_\theta,D}, \overline{\theta}_{n_\theta,D}] \mapsto \mathbb{R}^{n_\theta}$ are vector-valued functions that contain the relationships in eqs. (7) and eq. (8), respectively. Note that the dimension of vector $\boldsymbol{z}$ is $(n_x + n_\theta) \times 1$. With all the above definitions, the augmented failure probability in the standard normal space is cast as:

$$p_F^A = \int_{g_{\boldsymbol{z}}(\boldsymbol{z}) \leq 0} f_{\boldsymbol{Z}}(\boldsymbol{z}) d\boldsymbol{z} \tag{10}$$

where $f_{\boldsymbol{Z}}(\boldsymbol{z})$ is the standard normal probability density function in $n_x + n_\theta$ dimensions; and $g_{\boldsymbol{z}}(\boldsymbol{z})$ is the performance function in the standard normal space, which is defined as $g_{\boldsymbol{z}}(\boldsymbol{z}) = g\left(\boldsymbol{T}_{PX}\left(\boldsymbol{z}_P|\boldsymbol{T}_{A\Theta}(\boldsymbol{z}_A)\right)\right) = g\left(\boldsymbol{T}_{PX}\left(\boldsymbol{z}_P|\boldsymbol{\theta}\right)\right) = g(\boldsymbol{x})$, where $\boldsymbol{T}_{PX}$ and $\boldsymbol{T}_{A\Theta}$ denote the inverse vector-valued functions of $\boldsymbol{T}_{XP}$ and $\boldsymbol{T}_{\Theta A}$, respectively.

3.4. Approximation of the Failure Probability as Function of the Distribution Parameters

The augmented failure probability integral in eq. (10) can be estimated by means of the First-Order Reliability Method (FORM, see e.g. [32]). For that purpose, the first step is identifying the so-called design point $\boldsymbol{z}^*$, which is the realization of $\boldsymbol{z}$ with smallest Euclidean norm with respect to the origin of the standard normal space such that $g_{\boldsymbol{z}}(\boldsymbol{z}) = 0$. Methods for determining the design point are well documented in the literature, see e.g. [45, 46]. The well-known improved Hasofer-Lind-Rackwitz-Fiessler (iHLRF, [47]) is employed for determining the design point, as it exhibits an adequate performance for the class of problems considered in this work. The iHLRF algorithm demands repeated evaluations of both the performance function $g_{\boldsymbol{z}}(\boldsymbol{z})$ and its gradient $\nabla g_{\boldsymbol{z}}$. Details about the calculation of the gradient are discussed in Section 3.5.
Once the design point has been found, the next step associated with FORM is approximating the performance function $g_{\boldsymbol{z}}(\boldsymbol{z})$ about the design point $\boldsymbol{z}^*$ by means of a first-order Taylor expansion $g^L_{\boldsymbol{z}}(\boldsymbol{z})$. Recalling that $g_{\boldsymbol{z}}(\boldsymbol{z}^*) = 0$, this linear approximation of the performance function is:

$$g_{\boldsymbol{z}}(\boldsymbol{z}) \approx g^L_{\boldsymbol{z}}(\boldsymbol{z}) = \nabla g_{\boldsymbol{z}}(\boldsymbol{z}^*)^T \left(\boldsymbol{z} - \boldsymbol{z}^*\right). \tag{11}$$

where $(\cdot)^T$ denotes transpose of the argument. Taking into account the above approximation, it is possible to obtain an estimate of the augmented failure probability integral in eq. (10). However, it should be recalled that the objective is not estimating this augmented probability but instead, calculating the failure probability as a function of the distribution parameters. For that purpose, consider that:

$$p_F(\boldsymbol{\theta}) = P\left[g_{\boldsymbol{z}}(\boldsymbol{z}) \leq 0|\boldsymbol{\theta}\right] \tag{12}$$

where $P[\cdot]$ denotes probability of the term within brackets. The above equation indicates that $p_F(\boldsymbol{\theta})$ is the probability that the performance function in the standard normal space is equal or smaller than zero given a fixed value of the distribution parameters $\boldsymbol{\theta}$. Eq. (12) can be solved by considering the linear approximation of the performance function in eq. (11).

$$\begin{aligned} p_F(\boldsymbol{\theta}) &\approx P\left[g^L_{\boldsymbol{z}}(\boldsymbol{z}) \leq 0|\boldsymbol{\theta}\right] \\ &\approx P\left[\nabla g_{\boldsymbol{z}}(\boldsymbol{z}^*)^T \left(\boldsymbol{z} - \boldsymbol{z}^*\right) \leq 0|\boldsymbol{\theta}\right] \end{aligned} \tag{13}$$

This last expression can be further simplified by taking into account the following two issues. First, it is recalled from Section 3.3 that $\boldsymbol{z} = [\boldsymbol{z}_P^T, \boldsymbol{z}_A^T]^T$, which implies that the linear approximation of the performance function can be split into two parts associated with $\boldsymbol{z}_P$ and $\boldsymbol{z}_A$. Second, eq. (13) is conditioned on a particular value of the distribution parameters $\boldsymbol{\theta}$. According to Section 3.3, for each realization $\boldsymbol{\theta}$, there is a realization $\boldsymbol{z}_A$ which is given by eq. (8). This implies that in eq. (13), vector $\boldsymbol{z}_A$ assumes a specific value. With these two considerations, eq. (8) can be further simplified as:

$$\begin{aligned} p_F(\boldsymbol{\theta}) &\approx P\left[\nabla g_{\boldsymbol{z}_P}(\boldsymbol{z}_P^*)^T\left(\boldsymbol{z}_P - \boldsymbol{z}_P^*\right) + \nabla g_{\boldsymbol{z}_A}(\boldsymbol{z}_A^*)^T\left(\boldsymbol{z}_A - \boldsymbol{z}_A^*\right) \leq 0\right] \\ &\approx P\left[\nabla g_{\boldsymbol{z}_P}(\boldsymbol{z}_P^*)^T \boldsymbol{z}_P \leq \nabla g_{\boldsymbol{z}_P}(\boldsymbol{z}_P^*)^T \boldsymbol{z}_P^* - \nabla g_{\boldsymbol{z}_A}(\boldsymbol{z}_A^*)^T\left(\boldsymbol{z}_A - \boldsymbol{z}_A^*\right)\right] \end{aligned} \tag{14}$$

where $\boldsymbol{z}_P^*$ and $\boldsymbol{z}_A^*$ contain the first n_x and last n_θ entries of the design point $\boldsymbol{z}^*$, respectively; and where $\nabla g_{\boldsymbol{z}_P}(\boldsymbol{z}_P^*)$ and $\nabla g_{\boldsymbol{z}_A}(\boldsymbol{z}_A^*)$ contain the first n_x and last n_θ entries of the gradient of the performance function evaluated at design point $\nabla g_{\boldsymbol{z}}(\boldsymbol{z}^*)$, respectively. Note that in eq. (14), the dependence with respect to $\boldsymbol{\theta}$ is dropped as $\boldsymbol{z}_A$ fulfils that role. Eq. (14) can be further simplified by noting that $\nabla g_{\boldsymbol{z}_P}(\boldsymbol{z}_P^*)^T \boldsymbol{z}_P$ follows a normal distribution with zero mean and standard deviation $\|\nabla g_{\boldsymbol{z}_P}(\boldsymbol{z}_P^*)\|$, where $\|\cdot\|$ denotes Euclidean norm of the argument. Thus:

$$p_F(\boldsymbol{\theta}) \approx \Phi\left(\frac{\nabla g_{\boldsymbol{z}_P}(\boldsymbol{z}_P^*)^T \boldsymbol{z}_P^* - \nabla g_{\boldsymbol{z}_A}(\boldsymbol{z}_A^*)^T\left(\boldsymbol{T}_{\Theta A}(\boldsymbol{\theta}) - \boldsymbol{z}_A^*\right)}{\|\nabla g_{\boldsymbol{z}_P}(\boldsymbol{z}_P^*)\|}\right) \tag{15}$$

where $\Phi(\cdot)$ is the standard normal cumulative distribution function. In this last equation, the equality $\boldsymbol{z}_A = \boldsymbol{T}_{\Theta A}(\boldsymbol{\theta})$ has been introduced (see Section 3.3) to express the approximation in terms of $\boldsymbol{\theta}$.

The approximation in eq. (15) involves the design point $\boldsymbol{z}^*$ and the gradient of the performance function evaluated at the design point $\nabla g_{\boldsymbol{z}}(\boldsymbol{z}^*)$. Both quantities are obtained after determining the design point associated with the augmented reliability problem. It is noted that the approximation in eq. (15) is an explicit function of the distribution parameters $\boldsymbol{\theta}$, which can be evaluated at negligible numerical costs. Hence, this approximation can be plugged directly into eqs. (3) and (4) in order to estimate the membership function associated with the failure probability.

From the above discussion, it is noted that the proposed approach for coping with fuzzy failure probability can be extremely convenient from a numerical viewpoint. Indeed, only a single reliability analysis performed in the augmented reliability space suffices for constructing the explicit approximation of the failure probability shown in eq. (15). Nevertheless, it should be stated that this approximation can be relatively rough. Indeed, the approximation is based on a linearization of the performance function about the design point. Therefore, possible nonlinear behavior of the limit state surface may not be captured appropriately. Moreover, the linearization is performed in the standard normal space. This introduces another source of error, as transformations from physical space to standard normal are, in most cases, nonlinear. Hence, the proposed approach can be regarded as a trade off between accuracy of the approximation and decreased numerical efforts.

3.5. Implementation Aspects

3.5.1. Selection of Support for Auxiliary Probability Distributions

The auxiliary probability distribution associated with the l-th distribution parameter θ_l corresponds to a uniform one, that is $\Theta_l \sim \mathcal{U}[\underline{\theta}_{l,D}, \overline{\theta}_{l,D}]$, as already discussed in Section 3.2. In principle, the bounds $[\underline{\theta}_{l,D}, \overline{\theta}_{l,D}]$ for this auxiliary uniform distribution could be selected equal to the support $[\underline{\theta}_l, \overline{\theta}_l]$ of the fuzzy triangular set $\tilde{\theta}_l$. However, such selection may be not be appropriate: as the augmented reliability problem is solved in the standard normal space and taking into account eq. (8), it would happen that the bounds $[\underline{\theta}_{l,D}, \overline{\theta}_{l,D}]$ would be located at minus/plus infinity in the standard normal space. To avoid this situation, it is proposed to fix the location of the bounds $[\underline{\theta}_l, \overline{\theta}_l]$ of the fuzzy triangular set $\tilde{\theta}_l$ in the standard normal space at $\pm\beta_D$, where β_D is a real number. Such criterion is represented schematically in Figure 3, which illustrates the auxiliary uniform probability density function associated with θ_l. From this figure, it is noted that the probability content between $\underline{\theta}_{l,D}$ and $\underline{\theta}_l$ as well as the probability content between $\overline{\theta}_l$ and $\overline{\theta}_{l,D}$ is equal to $\Phi(-\beta_D)$ according to the aforementioned criterion.

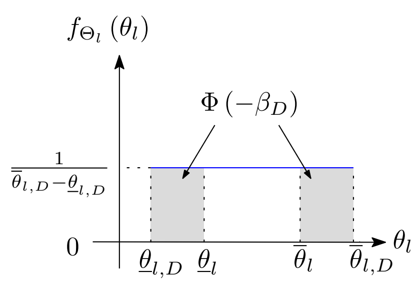

Figure 3: Schematic illustration of slope stability problem.

From a mathematical viewpoint, the proposed criterion implies fulfilling the follow set of equations (see eq. (8)).

$$\frac{\underline{\theta}_l - \underline{\theta}_{l,D}}{\overline{\theta}_{l,D} - \underline{\theta}_{l,D}} = \Phi(-\beta_D) \tag{16}$$

$$\frac{\overline{\theta}_l - \underline{\theta}_{l,D}}{\overline{\theta}_{l,D} - \underline{\theta}_{l,D}} = \Phi(+\beta_D) \tag{17}$$

The solution of the above equations allows determining the bounds $[\underline{\theta}_{l,D}, \overline{\theta}_{l,D}]$ for the auxiliary uniform distribution associated with θ_l. Numerical validation suggests that an appropriate value for β_D is 1.5.

As an additional remark, it should be noted that the proposed criterion for selecting the bounds of the auxiliary uniform probability distribution ensures that $\underline{\theta}_{l,D} < \underline{\theta}_l$ and $\overline{\theta}_{l,D} > \overline{\theta}_l$. A similar idea has been applied in [48].

3.5.2. Gradient of Performance Function

The identification of the design point associated with the augmented reliability problem demands evaluation of the gradient of the performance function in the standard normal space $\nabla g_{\boldsymbol{z}}$, as already discussed in Section 3.4. A convenient means for evaluating such gradient is using the associated Jacobian matrix $\boldsymbol{J}$, that is:

$$\nabla g_{\boldsymbol{z}}^T = \nabla g_{\boldsymbol{x}}^T \boldsymbol{J} \tag{18}$$

where $\nabla g_{\boldsymbol{x}}$ is the gradient of the performance function with respect to the input parameters $\boldsymbol{x}$ and whose dimension is $n_x \times 1$. The Jacobian matrix $\boldsymbol{J}$ possesses dimensions $n_x \times (n_x + n_\theta)$ and is defined as:

$$\boldsymbol{J} = \left(\begin{array}{ccc|ccc} \frac{\partial x_1}{\partial z_{P,1}} & \cdots & \frac{\partial x_1}{\partial z_{P,n_x}} & \frac{\partial x_1}{\partial z_{A,1}} & \cdots & \frac{\partial x_1}{\partial z_{A,n_\theta}} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial x_{n_x}}{\partial z_{P,1}} & \cdots & \frac{\partial x_{n_x}}{\partial z_{P,n_x}} & \frac{\partial x_{n_x}}{\partial z_{A,1}} & \cdots & \frac{\partial x_{n_x}}{\partial z_{A,n_\theta}} \end{array} \right) \tag{19}$$

The partial derivatives $\partial x_i/\partial z_{P,j},\ i,j = 1,\ldots,n_x$ can be obtained by differentiating eq. (7), leading to:

$$\frac{\partial x_i}{\partial z_{P,j}} = \begin{cases} \frac{\phi\left(z_{P,j}\right)}{f_{X_i}\left(x_i|\boldsymbol{\theta}^{(i)}\right)}, & \text{if } i=j \\ 0, & \text{otherwise} \end{cases},\ i,j = 1,\ldots,n_x, \tag{20}$$

where $\phi(\cdot)$ denotes the standard normal probability density function; while partial derivatives $\partial x_i/\partial z_{A,l},\ i = 1,\ldots,n_x,\ l = 1,\ldots,n_\theta$ can be obtained by differentiating eq. (8), leading to:

$$\frac{\partial x_i}{\partial z_{A,l}} = \begin{cases} -\frac{1}{f_{X_i}\left(x_i|\boldsymbol{\theta}^{(i)}\right)} \frac{\partial F_{X_i}\left(x_i|\boldsymbol{\theta}^{(i)}\right)}{\partial \theta_l} \left(\overline{\theta}_{l,D} - \underline{\theta}_{l,D}\right) \phi\left(z_l\right), & \text{if } \theta_l \in \boldsymbol{\theta}^{(i)} \\ 0, & \text{otherwise} \end{cases},\ i = 1,\ldots,n_x,\ l = 1,\ldots,n_\theta. \tag{21}$$

Several important issues should be noted from eqs. (18)–(21). First, from eq. (18), it is noted that the gradient in the standard normal space $\nabla g_{\boldsymbol{z}}$ possesses dimension $(n_x + n_\theta) \times 1$ while the gradient $\nabla g_{\boldsymbol{x}}$ possesses dimension $n_x \times 1$. In this sense, the gradient $\nabla g_{\boldsymbol{x}}$ expresses the rate of change of the performance function with respect to the input parameters. However, the gradient $\nabla g_{\boldsymbol{z}}$ measures the rate of change in the standard normal space, which is constructed based on the augmented reliability problem. The latter explains the different in dimensions of these two gradients. Second, the two gradients $\nabla g_{\boldsymbol{x}}$ and $\nabla g_{\boldsymbol{z}}$ are related between them by the Jacobian matrix $\boldsymbol{J}$. This matrix measures the rate of change of the input parameters $\boldsymbol{x}$ with respect to the coordinates $\boldsymbol{z}$ in the standard normal (and augmented) space. Third, the first n_x columns of this Jacobian are calculated using eq. (20). This equation appears when applying FORM in a purely probabilistic framework (see, e.g. [32]). Fourth, the last n_θ columns of the Jacobian matrix are calculated using eq. (21) and appear as a consequence of the augmented reliability problem. Fifth, closed-form expressions of eqs. (20) and (21) can be deduced for specific distributions.

From the above discussion, it is important to note that while this paper uses an augmented reliability approach, this does not significantly affect the calculation of the gradient of the performance function. In fact, only the gradient of the performance function with respect to the input parameters $\boldsymbol{x}$ is required, as in a standard FORM analysis. Therefore, the solution of the reliability problem in the augmented space which implies looking for the design point can be performed using standard algorithms (such as iHLRF) without significant modifications other than calculating the Jacobian matrix in eq. (19).

4. Examples

4.1. General Remarks

The performance of the proposed approach for solving problems of fuzzy probability analysis is illustrated in this Section by means of two examples. The first example involves a slab resting on a Winkler foundation, which is characterized considering a linear finite element model. The second example considers the stability analysis of a slope and involves a nonlinear finite element model. Both of these examples offer an excellent test bed, as validation calculations indicate that the First-Order Reliability Method is suitable for addressing them in a classical reliability setting (that is, without epistemic uncertainty) with sufficient accuracy.

In each of the two examples mentioned above, the performance of the augmented First-Order Reliability Method is compared with that of the classical First-Order Reliability Method in combination with the so-called vertex approach for coping with epistemic uncertainty. Comparisons with more advanced sampling approaches as those reported in, e.g. [21, 22, 49], etc., are not included in here, as the main purpose is illustrating how the First-Order Reliability method can be adapted with minimal modifications in order to estimate fuzzy probabilities.

4.2. Fuzzy Probability Analysis of a Slab Resting on a Winkler Foundation

This example is partially taken fom [50, 51]. It involves a simply supported slab that rests over a Winkler foundation. This slab supports a vertically uniformly distributed load pointing downwards. Figure 4 illustrates the slab.

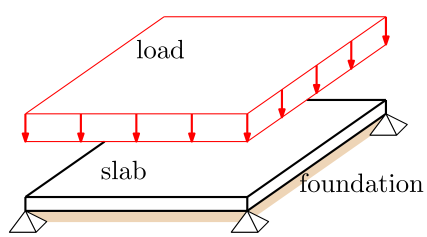

Figure 4: Slab resting on a Winkler foundation

The slab possesses a thickness of 20 [cm] and the length of its edges is 5 [m]. The Young's modulus and Poisson ration of the slab are $E = 2 \times 10^{10}$ [Pa] and $\nu = 0.2$, respectively. The modulus of the Winkler foundation (c_w) and the uniformly distributed load (w) are modeled as lognormal random variables whose distribution parameters are characterized as fuzzy variables with triangular membership as shown in Table 1.

Distribution parameter	Triangular membership
mean of c_w	$\langle 180, 200, 220 \rangle$ [MPa]
std. deviation of c_w	$\langle 18, 20, 22 \rangle$ [MPa]
mean of w	$\langle 9, 10, 11 \rangle$ [kPa]
std. deviation of w	$\langle 0.9, 1.0, 1.1 \rangle$ [kPa]

Table 1: Triangular membership functions associated with distribution parameters of c_w and w

The objective is calculating the membership function associated with the probability that the vertical displacement of the slab at its center point exceeds a threshold level of 0.8 [mm]. This displacement is determined by means of a linear finite element model comprising about 2700 degrees-of-freedom. The Winkler foundation is included by means of equivalent springs at the nodes of the finite element model [52]. The gradient of the performance function is calculated analytically and is a byproduct of a structural analysis [53].

The membership function of the failure probability $\mu_{\tilde{p}_F}(p_F)$ is calculated by means of α-level optimization considering 11 discrete levels. The approximate expression for estimating the failure probability as a function of the distribution parameters as presented in eq. (15) is considered for performing α-level optimization. The results obtained are shown in Figure 5 under the label 'aFORM'.

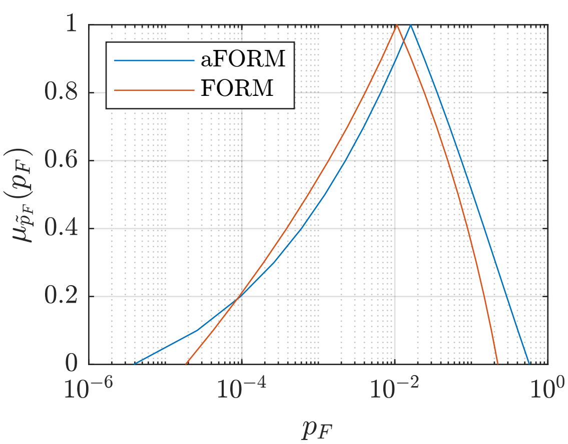

Figure 5: Membership function of failure probability p_F associated with slab over Winkler foundation

To validate the results produced with the proposed approach, the membership function is calculated using standard FORM to conduct reliability analysis combined with the vertex method (see, e.g. [41]) to analyze each α-level. The results obtained with this procedure are labeled as

'FORM' in Figure 5. It is observed that there is an overall good match between both membership functions. This is quite remarkable taking into account that the proposed approach (aFORM) demanded only 0.6% of the number of structural analyses required by the reference approach (FORM and vertex analysis).

4.3. Fuzzy Probability Analysis of Slope Stability

This example involves slope stability analysis. This problem is based on an example contained in [54] and is solved with the nonlinear finite element software provided in that reference. Figure 6 illustrates the physical dimensions of the problem.

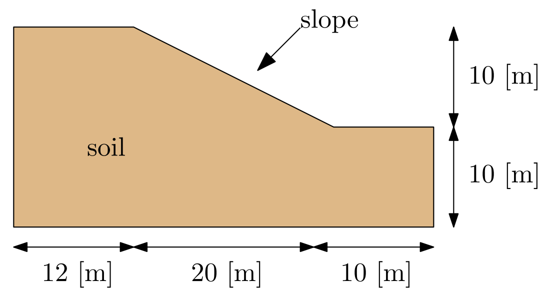

Figure 6: Schematic illustration of slope stability problem.

The soil is modeled as elastoplastic according to the Mohr-Coulomb criterion. The Young's modulus and Poisson ratio of the soil are taken as 100 [MPa] and 0.3, respectively, while its unit weight is 20 [kN/m^3]. The friction angle of the soil ψ and cohesion c are uncertain and are described considering lognormal random variables. The mean and standard deviation of each of these random variables are characterized as fuzzy variables with triangular membership functions. The specific values considered are summarized in Table 2.

Distribution parameter	Triangular membership
mean of ψ	$\langle 22°, 23°, 24° \rangle$
std. deviation of ψ	$\langle 2.2°, 2.3°, 2.4° \rangle$
mean of c	$\langle 4, 5, 6 \rangle$ [kPa]
std. deviation of c	$\langle 0.4, 0.5, 0.6 \rangle$ [kPa]

Table 2: Triangular membership functions associated with distribution parameters of ψ and c

According to results reported in the literature, friction angle and cohesion are negatively correlated [55]. Ignoring their correlation may lead to overestimation of the failure probability [56]. Nonetheless, it has also been reported that estimates of the failure probability are highly sensitive to the model considered for characterizing the dependence between these two parameters

[57]. Therefore, and for the sake of simplicity, the correlation between friction angle ψ and the cohesion c is disregarded in the following. This simplification is not detrimental with respect to illustrating the capabilities of the proposed approach.

The objective is determining the probability that the so-called factor of safety associated with the slope is smaller or equal than 1, as this separates a stable slope from an unstable one. In this context, the factor of safety is understood as the ratio of the soil shear strength to the shear stress of a possible sliding surface in the slope. Whenever the value of this safety factor is smaller than one, an undesirable behavior occurs. Therefore, the performance function is in this case:

$$g(\boldsymbol{x}) = \mathrm{FS}(\boldsymbol{x}) - 1 \tag{22}$$

where $\mathrm{FS}(\boldsymbol{x})$ represents the factor of safety as a function of the input parameters, where $\boldsymbol{x} = [\psi, c]^T$.

The gradient of the performance function $\nabla g_{\boldsymbol{x}}$ is estimated using finite differences [58].

Given that the uncertainty in the input parameters of the slope model are characterized using fuzzy probabilities, the failure probability becomes a fuzzy variable itself. The membership function of the failure probability $\mu_{\tilde{p}_F}(p_F)$ is characterized using the α-level optimization procedure. For this purpose, 11 discrete levels are considered. The proposed approximation of the failure probability as cast in eq. (15) is considered for carrying out the α-level optimization. The obtained membership function is reported in Figure 7 with the label 'aFORM'.

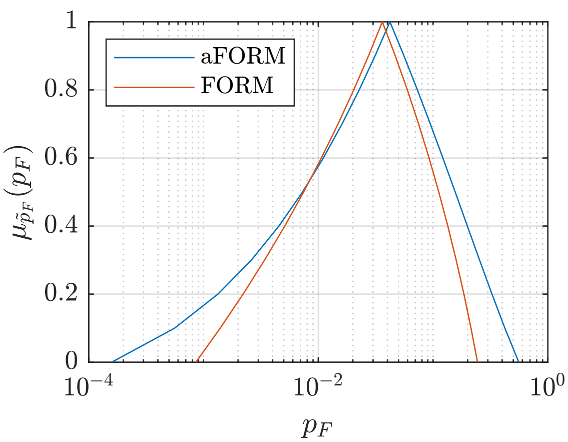

Figure 7: Membership function of failure probability p_F associated with slope stability problem

A reference membership function is then produced by combining standard FORM for reliability analysis with the vertex method for solving α-level optimization. The membership function obtained by such method is plotted in Figure 7 with the label 'FORM'. It is readily observed that both membership curves possess an overall good match, in the sense that the correct order of magnitude of the failure probability is predicted for different membership values. Of course, when comparing specific points of both membership curves, some differences are observed. Nonetheless, it should be kept in mind that the numerical efforts associated with the proposed approach (labeled as 'aFORM') are much lower than the reference one (labeled as 'FORM'). In fact, the proposed 'aFORM' approach demands performing a total of 12 finite element analyses. In comparison, the approach labeled as 'FORM' entails 2151 finite element analyses. This implies that the proposed approach demands only 0.56% of the numerical effort associated with the reference solution.

5. Conclusions and Outlook

This paper has presented an approach for estimating fuzzy failure probabilities. The approach is specifically targeted at problems where input parameters of a model are described through random variables while the associated distribution parameters are characterized as fuzzy sets. The core of the proposed approach lies in formulating an augmented reliability problem, where epistemic distribution parameters are characterized through auxiliary probability distributions. The resulting augmented reliability problem is solved using the First-Order Reliability Method. By using the information retrieved from this augmented reliability analysis, it is possible to generate an approximation of the failure probability which is an explicit function of the distribution parameters.

The results obtained from the example indicate that the membership function calculated using the proposed approach offers an overall reasonable match with reference results. In other words, the proposed approach provides a rough approximation of the membership function. However, such an approximation is obtained at reduced numerical costs, as a single reliability analysis suffices for determining the complete membership function. Furthermore, numerical results indicate that failure probabilities are highly sensitive with respect to epistemic uncertainty associated with distribution parameters. In fact, variations of the failure probability in orders of magnitude were observed in both examples. This highlights the relevance of conducting the class of analysis proposed in this work, as it may reveal important information on the sensitivity of the failure probability. Moreover, the approach implemented in this work requires minimal modifications with respect to the well-established First-Order Reliability Method. This constitutes an important advantage, as the approach can be easily integrated with existing numerical algorithms.

Future research efforts will aim at extending the range of application of the proposed approach. A path for development would involve constructing multiple approximations of the failure probability instead of a single one. Such an approach could be most useful for capturing possible nonlinear behavior of the limit state surface. Indeed, individual linear approximations can be combined in order to produce a nonlinear approximation with improved quality (see, e.g. [59, 60]). Undoubtedly, this nonlinear approximation could lead to a better estimation of the membership function associated with the failure probability.

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Appendix A. Transformation to Standard Normal Space for Normal and Lognormal Random Variables

Closed-form expressions for eqs. (7) and (8) can be deduced for several types of random variables. For example, assuming that $n_x = 1$ and that X represents a normal distribution with distribution parameters $\boldsymbol{\theta} = [\mu,\ \sigma]$, where μ and σ denote mean and standard deviation, it is possible to deduce that:

$$\begin{bmatrix} z_P \\ z_{A,1} \\ z_{A,2} \end{bmatrix} = \begin{bmatrix} \frac{x-\mu}{\sigma} \\ \Phi^{-1}\left(\frac{\mu-\underline{\mu}_D}{\overline{\mu}_D-\underline{\mu}_D}\right) \\ \Phi^{-1}\left(\frac{\sigma-\underline{\sigma}_D}{\overline{\sigma}_D-\underline{\sigma}_D}\right) \end{bmatrix}. \tag{A.1}$$

Assuming that $n_x = 1$ and that X represents a lognormal distribution with distribution parameters $\boldsymbol{\theta} = [\mu,\ \sigma]$, where μ and σ denote mean and standard deviation, it is possible to deduce that:

$$\begin{bmatrix} z_P \\ z_{A,1} \\ z_{A,2} \end{bmatrix} = \begin{bmatrix} \frac{\ln(x)-\mu_G(\mu,\sigma)}{\sigma_G(\mu,\sigma)} \\ \Phi^{-1}\left(\frac{\mu-\underline{\mu}_D}{\overline{\mu}_D-\underline{\mu}_D}\right) \\ \Phi^{-1}\left(\frac{\sigma-\underline{\sigma}_D}{\overline{\sigma}_D-\underline{\sigma}_D}\right) \end{bmatrix} \tag{A.2}$$

where μ_G and σ_G are defined as:

$$\mu_G(\mu, \sigma) = \ln\left(\frac{\mu^2}{\sqrt{\mu^2 + \sigma^2}}\right) \tag{A.3}$$

$$\sigma_G(\mu, \sigma) = \sqrt{\ln\left(1 + \frac{\sigma^2}{\mu^2}\right)}. \tag{A.4}$$

References

[1] A. Ang, W. Tang, Probability Concepts in Engineering: Emphasis on Applications to Civil and Environmental Engineering, Wiley, 2007.

[2] R. Melchers, A. Beck, Structural Reliability Analysis and Prediction, 3rd Edition, John Wiley & Sons, 2018. doi:10.1002/9781119266105.

[3] M. Hohenbichler, R. Rackwitz, First-order concepts in system reliability, Structural Safety 1 (3) (1983) 177–188.

[4] K. Breitung, Asymptotic approximations for multinormal integrals, Journal of Engineering Mechanics 110 (3) (1984) 357–366. doi:10.1061/(ASCE)0733-9399(1984)110:3(357).

[5] B. Echard, N. Gayton, M. Lemaire, AK-MCS: An active learning reliability method combining Kriging and Monte Carlo simulation, Structural Safety 33 (2) (2011) 145 – 154. doi:https://doi.org/10.1016/j.strusafe.2011.01.002.
URL http://www.sciencedirect.com/science/article/pii/S0167473011000038

[6] S. Marelli, B. Sudret, An active-learning algorithm that combines sparse polynomial chaos expansions and bootstrap for structural reliability analysis, Structural Safety 75 (2018) 67 – 74. doi:https://doi.org/10.1016/j.strusafe.2018.06.003.
URL http://www.sciencedirect.com/science/article/pii/S0167473017302977

[7] J. Li, J. Chen, Probability density evolution method for dynamic response analysis of structures with uncertain parameters, Computational Mechanics 34 (5) (2004) 400–409. doi:10.1007/s00466-004-0583-8.
URL https://doi.org/10.1007/s00466-004-0583-8

[8] G. Chen, D. Yang, Direct probability integral method for stochastic response analysis of static and dynamic structural systems, Computer Methods in Applied Mechanics and Engineering 357 (2019) 112612. doi:https://doi.org/10.1016/j.cma.2019.112612.
URL http://www.sciencedirect.com/science/article/pii/S0045782519304888

[9] H. Vanvinckenroye, I. Kougioumtzoglou, V. Denoël, Reliability function determination of nonlinear oscillators under evolutionary stochastic excitation via a galerkin projection technique, Nonlinear Dynamics 95 (1) (2019) 293–308. doi:10.1007/s11071-018-4564-8.
URL https://doi.org/10.1007/s11071-018-4564-8

[10] I. Papaioannou, W. Betz, K. Zwirglmaier, D. Straub, MCMC algorithms for subset simulation, Probabilistic Engineering Mechanics 41 (2015) 89–103. doi:http://dx.doi.org/10.1016/j.probengmech.2015.06.006.
URL http://www.sciencedirect.com/science/article/pii/S0266892015300205

[11] S.-K. Au, E. Patelli, Rare event simulation in finite-infinite dimensional space, Reliability Engineering & System Safety 148 (2016) 67 – 77. doi:http://dx.doi.org/10.1016/j.ress.2015.11.012.
URL http://www.sciencedirect.com/science/article/pii/S0951832015003397

[12] A. Der Kiureghian, O. Ditlevsen, Aleatory or epistemic? does it matter?, Structural Safety 31 (2) (2009) 105–112.

[13] M. Faes, D. Moens, Recent trends in the modeling and quantification of non-probabilistic uncertainty, Archives of Computational Methods in Engineering 27 (3) (2020) 633–671.
URL https://doi.org/10.1007/s11831-019-09327-x

[14] S. Zschocke, F. Leichsenring, W. Graf, M. Kaliske, A concept for data-driven computational mechanics in the presence of polymorphic uncertain properties, Engineering Structures 267 (2022) 114672. doi:https://doi.org/10.1016/j.engstruct.2022.114672.
URL https://www.sciencedirect.com/science/article/pii/S0141029622007660

[15] M. Beer, S. Ferson, V. Kreinovich, Imprecise probabilities in engineering analyses, Mechanical Systems and Signal Processing 37 (1-2) (2013) 4–29. doi:http://dx.doi.org/10.1016/j.ymssp.2013.01.024.
URL http://www.sciencedirect.com/science/article/pii/S0888327013000812

[16] M. Faes, M. Daub, S. Marelli, E. Patelli, M. Beer, Engineering analysis with probability boxes: A review on computational methods, Structural Safety 93 (2021) 102092. doi:https://doi.org/10.1016/j.strusafe.2021.102092.
URL https://www.sciencedirect.com/science/article/pii/S0167473021000187

[17] B. Möller, M. Beer, Fuzzy Randomness, Springer Berlin Heidelberg, 2004. doi:10.1007/978-3-662-07358-2.

[18] X. Liu, Z. Kuang, L. Yin, L. Hu, Structural reliability analysis based on probability and probability box hybrid model, Structural Safety 68 (2017) 73–84. doi:https://doi.org/10.1016/j.strusafe.2017.06.002.
URL http://www.sciencedirect.com/science/article/pii/S0167473016301734

[19] C. Wang, H. Zhang, M. Beer, Computing tight bounds of structural reliability under imprecise probabilistic information, Computers & Structures 208 (2018) 92 – 104. doi:https://doi.org/10.1016/j.compstruc.2018.07.003.
URL http://www.sciencedirect.com/science/article/pii/S0045794918304267

[20] D. Alvarez, J. Hurtado, An efficient method for the estimation of structural reliability intervals with random sets, dependence modeling and uncertain inputs, Computers & Structures 142 (2014) 54–63. doi:http://dx.doi.org/10.1016/j.compstruc.2014.07.006.
URL http://www.sciencedirect.com/science/article/pii/S0045794914001503

[21] M. de Angelis, E. Patelli, M. Beer, Advanced Line Sampling for efficient robust reliability analysis, Structural Safety 52, Part B (2015) 170–182. doi:http://dx.doi.org/10.1016/j.strusafe.2014.10.002.
URL http://www.sciencedirect.com/science/article/pii/S0167473014000927

[22] M. Troffaes, Imprecise Monte Carlo simulation and iterative importance sampling for the estimation of lower previsions, International Journal of Approximate Reasoning 101 (2018) 31 – 48. doi:https://doi.org/10.1016/j.ijar.2018.06.009.
URL http://www.sciencedirect.com/science/article/pii/S0888613X17305868

[23] P. Wei, J. Song, S. Bi, M. Broggi, M. Beer, Z. Lu, Z. Yue, Non-intrusive stochastic analysis with parameterized imprecise probability models: II. reliability and rare

events analysis, Mechanical Systems and Signal Processing 126 (2019) 227 – 247. doi:https://doi.org/10.1016/j.ymssp.2019.02.015.
URL http://www.sciencedirect.com/science/article/pii/S0888327019300986

[24] H. Zhang, Interval importance sampling method for finite element-based structural reliability assessment under parameter uncertainties, Structural Safety 38 (2012) 1–10. doi:10.1016/j.strusafe.2012.01.003.

[25] X. Yuan, M. Faes, S. Liu, M. Valdebenito, M. Beer, Efficient imprecise reliability analysis using the augmented space integral, Reliability Engineering & System Safety 210 (2021) 107477. doi:https://doi.org/10.1016/j.ress.2021.107477.
URL https://www.sciencedirect.com/science/article/pii/S0951832021000454

[26] N.-C. Xiao, H.-Z. Huang, Z. Wang, Y. Pang, L. He, Reliability sensitivity analysis for structural systems in interval probability form, Structural and Multidisciplinary Optimization 44 (5) (2011) 691–705.

[27] W. Graf, M. Götz, M. Kaliske, Analysis of dynamical processes under consideration of polymorphic uncertainty, Structural Safety 52, Part B (2015) 194–201, Engineering Analyses with Vague and Imprecise Information. doi:http://dx.doi.org/10.1016/j.strusafe.2014.09.003.
URL http://www.sciencedirect.com/science/article/pii/S0167473014000861

[28] R. Schöbi, B. Sudret, Structural reliability analysis for p-boxes using multi-level meta-models, Probabilistic Engineering Mechanics 48 (2017) 27–38. doi:https://doi.org/10.1016/j.probengmech.2017.04.001.
URL http://www.sciencedirect.com/science/article/pii/S0266892017300152

[29] M. Faes, M. Valdebenito, D. Moens, M. Beer, Bounding the first excursion probability of linear structures subjected to imprecise stochastic loading, Computers & Structures 239 (2020) 106320. doi:https://doi.org/10.1016/j.compstruc.2020.106320.
URL http://www.sciencedirect.com/science/article/pii/S0045794920301231

[30] H. Zhang, R. Mullen, R. Muhanna, Interval monte carlo methods for structural reliability, Structural Safety 32 (3) (2010) 183–190. doi:https://doi.org/10.1016/j.strusafe.2010.01.001.
URL https://www.sciencedirect.com/science/article/pii/S0167473010000123

[31] S. Au, Reliability-based design sensitivity by efficient simulation, Computers & Structures 83 (14) (2005) 1048–1061.

[32] A. Der Kiureghian, Engineering Design Reliability Handbook, CRC Press, 2004, Ch. First- and Second-Order Reliability Methods.

[33] L. Katafygiotis, K. Zuev, Geometric insight into the challenges of solving high-dimensional reliability problems, Probabilistic Engineering Mechanics 23 (2-3) (2008) 208–218.

[34] X. Du, Unified uncertainty analysis by the First Order Reliability Method, Journal of Mechanical Design 130 (9), 091401 (08 2008). arXiv:https://asmedigitalcollection.asme.org/mechanicaldesign/article-pdf/130/9/091401/5547960/091401_1.pdf, doi:10.1115/1.2943295.
URL `https://doi.org/10.1115/1.2943295`

[35] W. Yao, X. Chen, Y. Huang, M. van Tooren, An enhanced unified uncertainty analysis approach based on first order reliability method with single-level optimization, Reliability Engineering & System Safety 116 (2013) 28–37. doi:10.1016/j.ress.2013.02.014.

[36] X. Liu, L. Yin, L. Hu, Z. Zhang, An efficient reliability analysis approach for structure based on probability and probability box models, Structural and Multidisciplinary Optimization 56 (1) (2017) 167–181. doi:10.1007/s00158-017-1659-7.
URL `http://dx.doi.org/10.1007/s00158-017-1659-7`

[37] U. Alibrandi, C. Koh, First-order reliability method for structural reliability analysis in the presence of random and interval variables, ASCE-ASME J Risk and Uncert in Engrg Sys Part B Mech Engrg 1 (4), 041006 (10 2015). arXiv:https://asmedigitalcollection.asme.org/risk/article-pdf/1/4/041006/6071600/risk_1_4_041006.pdf, doi:10.1115/1.4030911.
URL `https://doi.org/10.1115/1.4030911`

[38] K. Bathe, Finite Element Procedures, Prentice Hall, New Jersey, 1996.

[39] C. Bucher, Computational Analysis of Randomness in Structural Mechanics, CRC Press, 2009.

[40] X. Yuan, S. Liu, M. Valdebenito, J. Gu, M. Beer, Efficient procedure for failure probability function estimation in augmented space, Structural Safety 92 (2021) 102104. doi:https://doi.org/10.1016/j.strusafe.2021.102104.
URL https://www.sciencedirect.com/science/article/pii/S016747302100028X

[41] D. Moens, M. Hanss, Non-probabilistic finite element analysis for parametric uncertainty treatment in applied mechanics: Recent advances, Finite Elements in Analysis and Design 47 (1) (2011) 4–16. doi:10.1016/j.finel.2010.07.010.

[42] M. Hanss, Applied Fuzzy Arithmetic, Springer Berlin Heidelberg, 2005. doi:10.1007/b138914.

[43] M. Beer, Uncertain structural design based on nonlinear fuzzy analysis, Journal of Applied Mathematics and Mechanics (ZAMM) 84 (10-11) (2004) 740–753. doi:10.1002/zamm.200410154.

[44] J. Ching, Y. Hsieh, Local estimation of failure probability function and its confidence interval with maximum entropy principle, Probabilistic Engineering Mechanics 22 (1) (2007) 39–49.

[45] R. Rackwitz, B. Fiessler, Structural reliability under combined random load sequences, Computers & Structures 9 (5) (1978) 489 – 494. doi:https://doi.org/10.1016/0045-7949(78)90046-9.
URL http://www.sciencedirect.com/science/article/pii/0045794978900469

[46] S. Santos, L. Matioli, A. Beck, New optimization algorithms for structural reliability analysis, Computer Modeling in Engineering & Sciences(CMES) 83 (1) (2012) 23–55.

[47] Y. Zhang, A. Der Kiureghian, Two improved algorithms for reliability analysis, in: Reliability and Optimization of Structural Systems: Proceedings of the sixth IFIP WG7. 5 working conference on reliability and optimization of structural systems 1994, Springer, 1995, pp. 297–304.

[48] J. Song, P. Wei, M. Valdebenito, S. Bi, M. Broggi, M. Beer, Z. Lei, Generalization of non-intrusive imprecise stochastic simulation for mixed uncertain variables, Mechanical Systems and Signal Processing 134 (2019) 106316. doi:https://doi.org/10.1016/j.ymssp.2019.106316.
URL http://www.sciencedirect.com/science/article/pii/S0888327019305370

[49] P. Wei, Z. Lu, J. Song, Extended Monte Carlo simulation for parametric global sensitivity analysis and optimization, AIAA Journal 52 (2014) 867–878. doi:10.2514/1.J052726.

[50] J. Hurtado, D. Alvarez, J. Ramirez, Fuzzy structural analysis based on fundamental reliability concepts, Computers & Structures 112–113 (2012) 183–192. doi:10.1016/j.compstruc.2012.08.004.

[51] M. Valdebenito, C. Pérez, H. Jensen, M. Beer, Approximate fuzzy analysis of linear structural systems applying intervening variables, Computers & Structures 162 (162) (2016) 116–129. doi:10.1016/j.compstruc.2015.08.020.

[52] E. Oñate, Structural Analysis with the Finite Element Method. Linear Statics. Volume 2. Beams, Plates and Shells, Springer, 2013.

[53] R. Haftka, Z. Gürdal, Elements of Structural Optimization, 3^{rd} Edition, Kluwer, Dordrecht, The Netherlands, 1992.

[54] I. Smith, D. Griffiths, L. Margetts, Programming the Finite Element Method, 5th Edition, John Wiley & Sons, Ltd, 2014. doi:10.1002/9781119189237.

[55] Y. Wang, O. Akeju, Quantifying the cross-correlation between effective cohesion and friction angle of soil from limited site-specific data, Soils and Foundations 56 (6) (2016) 1055–1070. doi:https://doi.org/10.1016/j.sandf.2016.11.009.
URL https://www.sciencedirect.com/science/article/pii/S0038080616301263

[56] S. Jiang, D. Li, Z. Cao, C. Zhou, K. Phoon, Efficient system reliability analysis of slope stability in spatially variable soils using Monte Carlo simulation, Journal of Geotechnical and Geoenvironmental Engineering 141 (2) (2015) 04014096. arXiv:http://dx.doi.org/10.1061/(ASCE)GT.1943-5606.0001227, doi:10.1061/(ASCE)GT.1943-5606.0001227.
URL http://dx.doi.org/10.1061/(ASCE)GT.1943-5606.0001227

[57] X.-S. Tang, D.-Q. Li, G. Rong, K.-K. Phoon, C.-B. Zhou, Impact of copula selection on geotechnical reliability under incomplete probability information, Computers and Geotechnics 49 (2013) 264–278. doi:10.1016/j.compgeo.2012.12.002.

[58] R. Burden, J. Faires, Numerical Analysis, Brooks Cole, 2010.

[59] X. Yuan, Y. Qian, J. Chen, M. Faes, M. Valdebenito, M. Beer, Global failure probability function estimation based on an adaptive strategy and combination algorithm, Reliability Engineering & System Safety 231 (2023) 108937. doi:https://doi.org/10.1016/j.ress.2022.108937. URL https://www.sciencedirect.com/science/article/pii/S095183202200552X

[60] I. Papaioannou, D. Straub, Combination line sampling for structural reliability analysis, Structural Safety 88 (2021) 102025. doi:https://doi.org/10.1016/j.strusafe.2020.102025. URL http://www.sciencedirect.com/science/article/pii/S0167473020301041